Iterated Function Systems of Interval Maps

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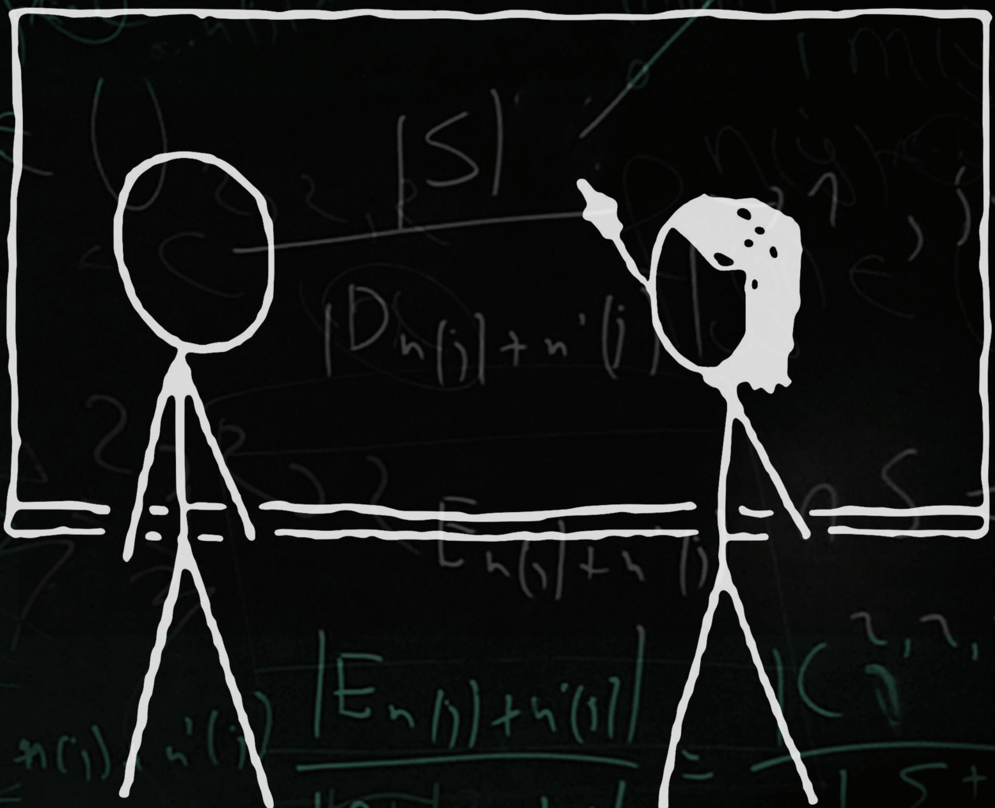

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Chapter 1

Introduction

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.

Henri Poincaré

The mathematical concept of dynamical system is a formalization to describe the evolution of points in a state space according to a given rule. The evolution rule specifies the next state or states of the current values of the state space. Mathematical models that describe the motion of planets, swinging of a clock pendulum, the flow of water in a pipe, or the number of fish each springtime in a lake include examples of dynamical systems. For instance the state of a swinging pendulum is its angle and angular velocity, and the evolution rule is Newton's equation $F = ma$.

The field of dynamical systems came to be in the 1600s by Newtons developments in differential equations. A couple of centuries later, Henri Poincaré realized that finding an exact solution for some dynamical systems, such as the three-body problem or describing the exact motion of three planetary bodies confined to the laws of gravitation, was essentially impossible. In fact the modern theory of dynamical systems originated from his work on the three-body problem of celestial mechanics. Poincaré developed a novel point of view to answer qualitative questions about the system rather than developing an exact quantitative solution.

Until the mid-1900s, dynamics was largely concerned with nonlinear oscillating systems and their applications to physics. The development of high speed computing led Lorenz to discover chaotic motion in dynamics in 1963. Since then interest in dynamics and chaos and its applications to real world systems has been growing. Much of modern research in dynamical systems is focused on the study of systems that arise in physics, astronomy, engineering, biology, chemistry, economics and other disciplines.

There are two main types of dynamical systems: differential equations and iterated maps. Differential equations describe the evolution of systems in continuous time like swinging of a clock pendulum, while iterated maps deal with discrete time like the number of bees each year. The theory of dynamical systems brings together many ideas from different branches of mathematics. It comprises a broad range of analytical, geometrical, topological, and numerical methods for analyzing systems.

As explained before a dynamical system consists of a state space, a set of times and a rule for the evolution. An evolution rule is deterministic if each state has a unique successor, and is stochastic or random if there is more than one possible successor for a given state. For instance the idealized coin toss has two successors with equal probability for each initial state. Generally speaking, an iterated function system, IFS, is a discrete time random dynamical system where the maps are drawn randomly from a collection of maps.

A deterministic system with discrete time is defined by a map $f : X \to X$ where X is the state space. The nth iterate of f is $f^n = f \circ \ldots \circ f$. Starting from the point x_0 in X at time 0, $x_n = f^n(x_0)$ represents its position at time n. The sequence of points $x_0, f(x_0), f^2(x_0), \ldots$ is the forward orbit of x_0. If $f^n(x_0) = x_0$ for $n = 1$ or some $n > 1$ then x_0 is a fixed point or a periodic point respectively. Deterministic evolution rules are invertible if each state has a unique precedent or preimage. In this case $x_n = f^n(x_0)$ can extend in both directions of time and the bi-infinite sequence x_n is called the full orbit of the initial value x_0. For an IFS there is a collection of functions $f_i : X \to X$ where $x_{n+1} = f_i(x_n)$, i is selected from a probability distribution.

This thesis contains three related articles focused on IFSs given by randomly iterating a finite number of continuous maps which act on the unit interval $[0, 1]$. They are inserted in separate chapters. There is no modification in the contents of the papers except small corrections such as typing errors and notational changes. Since the chapters are self contained it is possible to read each of them independently. Some repetition in content has been unavoidable. To prevent more repetition in technical concepts this chapter is confined to a short introduction of some basic and general concepts as well as motivation.

1.1 Discrete one dimensional dynamical systems

By a discrete one dimensional dynamical system we mean that the map f is a (typically smooth) function which maps a one dimensional space, like a circle or an interval, to itself.

Iterating one dimensional maps has a very long history. In order to construct an accurate calendar, the Babylonians had to consider a rotation of the circle and give a precise estimate for its angle of rotation based on a piece of its orbit. For this they, and later the Greeks, considered the line $(t, t\alpha)$ in the plane with slope α. To estimate α they developed a continued fraction algorithm. Ever since, continued fractions have played an important role in mathematics and in particular in number theory. Poincaré generalized this point of view by considering the dynamics of more general maps of the circle. A detailed understanding of the dynamics of these maps leads to small-divisor problems and plays an important role in the modern theory of celestial mechanics. Since the 18th century the iteration scheme to determine the zeros of a function has been another important one dimensional dynamical system.

Besides these historical motivations there are many reasons to study one dimensional dynamical systems. We list a few reasons here and refer to [25] for more.

- The dynamics of a one dimensional dynamical system can be very rich even for simple maps as polynomials. They serve as paradigm examples to lead the way for a study of higher dimensional systems.
- Considering dynamics with one dimensional state space allows us to use tools which are not available in higher dimensions, enabling more detailed statements.
- One dimensional dynamical systems display features of higher dimensional dynamics. For example aspects of the study of Lorenz flows in dimension 3 can be reduced to a class of maps on the interval. Aspects of the dynamics of Hénon maps are reminiscent to those of logistic maps; renormalization is the technique that provides a connection.

In the 1960s, Sharkovsky began to study the dynamics given by a continuous map on an interval, focusing in particular on the coexistence of periodic points of various periods, which is ruled by Sharkovsky's order. At present time there are many works dealing with dynamics of interval maps, considering both continuous and discontinuous maps, see for example [25, 26].

Example 1.1.1. *A paradigm example of interval dynamics is given by logistic maps*

$$f_\rho(x) = \rho x(1-x)$$

that have been proposed as toy models for population growth. Consider the logistic map on the interval $X = [0,1]$ *for parameter values* $0 < \rho \leq 4$*. If the parameter* ρ *is small enough, then all the trajectories converge to a fixed point: the population stabilizes. However, for bigger values of* ρ*, the dynamics may become very complicated.*

It is known that for an open, dense set of parameter values, f_ρ *possesses a unique periodic attractor whose basin of attraction has full Lebesgue measure [11]. This set of parameter values with periodic attractors is not of full Lebesgue measure in parameter space, even though it is open and dense: there is a set of parameters of positive Lebesgue measure for which there is no periodic attractor. In a precise sense, for Lebesgue almost every parameter value the dynamics is either* *regular or stochastic [22]. Regular means the existence of a periodic attractor, while stochastic stands for chaotic dynamics characterized by the existence of an absolutely continuous invariant measure.*

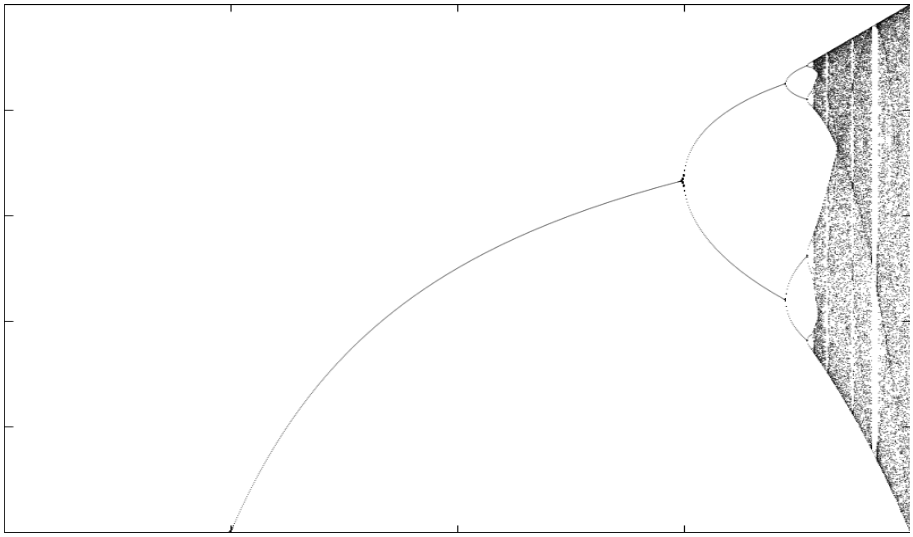

Figure 1.1: *A numerically computed bifurcation diagram of the logistic family $f_\rho(x) = \rho x(1-x)$. The horizontal axis is the parameter axis, running from $\rho = 0$ to $\rho = 4$. The vertical axis shows numerically computed attractors of logistic maps for given parameter values. Visible are intervals, also called windows, with periodic attractors.*

1.2 Iterated function systems

In an iterated function system one is given a collection of several maps which are used for iteration. The notion of iterated function system was introduced by Barnsley and Demko [5], but the concept is usually attributed to Hutchinson [13]. According to Vrscay [33] the idea is traced further back to Williams [34], who studied fixed points of finite compositions of contractive maps.

Iterated function systems are interacting with many fields of mathematics. For example they are useful for creating fractals, learning models, interesting probability distributions and analyzing stochastic processes with Markovian properties. It has been a very active topic of research due to its wide applications. See for example Barnsley [4] and Falconer [9] for more on connections to the theory of fractals, Jorgensen [16] for connections to representation theory and the theory of wavelets, Pesin [28] and Keller [19] for connections to the theory of (deterministic) dynamical systems, Stenflo [32] for a survey of place dependent random iterations and connections to the Ruelle-Perron-Frobenius theorem of statistical mechanics, Kaijser [17] for connections to the theory of products of random matrices, Roberts and Rosenthal [30] for a survey of Markov Chain Monte Carlo algorithms, Iosifescu and Grigorescu [15] for connections to the theory of continued fractions.

Iterated function systems can already generate interesting dynamics when the functions are contractive maps acting on complete metric spaces. In this case the orbits are often attracted to some fractal set.

1.2.1 Fractals

Suppose we are going to model objects like clouds, grass or plants. What could we use to define the geometries of these objects? Although polynomials can easily define objects with smooth geometry, they are pretty useless if we want to model complex objects that possess highly structured geometries. It was the fundamental work of Benoit Mandelbrot [23] that opened up a new way to model natural phenomena by describing fractals. Mandelbrot refers to the word fractal as an object who possesses self similarity, that is, it exhibits a similar pattern no matter how much you zoom in on the object.

Hutchinson developed a theory for fractals and probability measures supported on fractals [13]. He showed that a large class of fractals can be described in terms of iterations of multiple functions. Many know examples of fractals are attractors of appropriate IFSs. One of the best known and most easily constructed fractals is the well known middle third Cantor set that is constructed by a sequence of deletion operations on the unit interval.

Example 1.2.1. *Take $X = [0,1]$ and consider the two affine contractions*

$$\begin{aligned} f_1(x) &= \frac{x}{3}, \\ f_2(x) &= \frac{x}{3} + \frac{2}{3} \end{aligned}$$

on X. For a subset $Y \subset X$, let

$$S(Y) = f_1(Y) \cup f_2(Y).$$

The sets $S^n(X)$ form a decreasing sequence of compact sets. The middle third Cantor set is the limit

$$\lim_{n\to\infty} S^n(X).$$

There are many other examples of fractals with often fancy names such as Sierpinski gasket, Sierpinski carpet, Koch snowflake, Heighway dragon, Lévy dragon, McWorter pentigree, Pythagorean tree.

Figure 1.2: *The first steps in the construction of the middle third Cantor set, from top to bottom* $X, S(X), S^2(X), S^3(X)$.

1.2.2 Dynamical systems aspects of iterated function systems

Consider a complete metric space X and a finite number of continuous maps $f_i : X \to X$, $1 \le i \le k$. Take random compositions of these maps: for each iterate pick a symbol i at random, independently from previous iterates and with fixed probability $p_i > 0$, and then apply the map f_i. Given a sequence of outcomes $\omega_0, \ldots, \omega_{n-1}$, we thus find a composition $f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}$. We denote this composition by f_ω^n.

Especially for questions related to dynamical systems it is useful to study such random processes in the framework of skew product systems. Collect all sequences of symbols $1, 2, \ldots, k$ in the product set $\Sigma_k^+ = \{1, \ldots, k\}^{\mathbb{N}}$. Define $F^+ : \Sigma_k^+ \times X \to \Sigma_k^+ \times X$ by

$$F^+(\omega, x) = (\sigma\omega, f_{\omega_0}(x)).$$

Here $\sigma : \Sigma_k^+ \to \Sigma_k^+$ is the shift operator; $(\sigma\omega)_i = \omega_{i+1}$ for $\omega = (\omega_i)_0^\infty$. These step skew product systems provide a setting to study all possible compositions of the maps $f_1, \ldots, f_k$ in a single framework. Indeed, for initial conditions $(\omega, x) \in \Sigma_k^+ \times X$, the coordinate in X iterates as

$$x,\ f_{\omega_0}(x),\ f_{\omega_1} \circ f_{\omega_0}(x),\ f_{\omega_2} \circ f_{\omega_1} \circ f_{\omega_0}(x),\ \ldots \tag{1.2.1}$$

One can also consider two sided sequences of symbols in $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$ and the map $F : \Sigma_k \times X \to \Sigma_k \times X$ given by the same formula

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x))$$

as F^+.

Invariant measures are central in the study of these skew product systems. A natural measure on Σ_k^+ is the product measure ν^+, given probabilities p_i for symbols i. For instance the set of all sequences that start with the symbol 1 and are followed by the symbol 2, has product measure $p_1 p_2$. On Σ_k one likewise defines a product measure, which we denote by ν. These measures are invariant under the shift, so for instance $\nu^+(A) = \nu^+(\sigma^{-1}(A))$ for all measurable sets. Of particular importance are invariant product measures $\nu^+ \times m$ for F^+. The measure m here

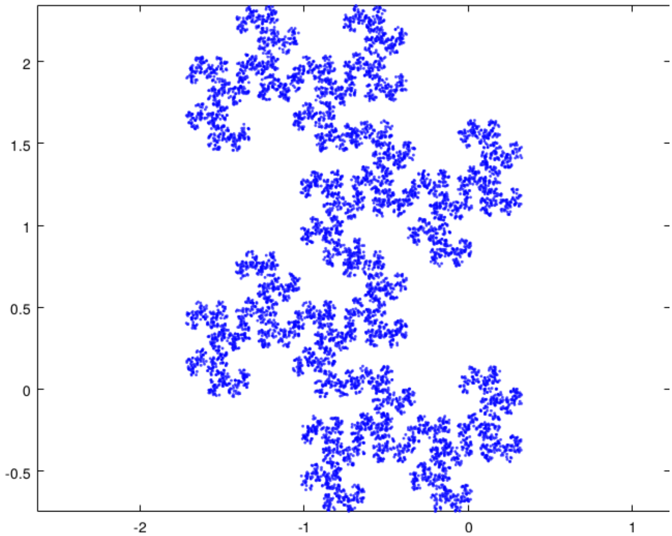

Figure 1.3: *A fractal in the plane that arises as the invariant set for the two affine contractions* $f_1(x,y) = \frac{1}{3}\sqrt{2}(x-y, x+y)$ *and* $f_2(x,y) = \frac{1}{3}\sqrt{2}(x-y, x+y) + (0, \frac{3}{2})$*. The fractal equals the limit* $\lim_{n\to\infty} S^n(X)$ *where* X *is a sufficiently large box around the origin and* $S(Y) = f_1(Y) \cup f_2(Y)$ *for subsets* $Y \subset X$*.*

is a stationary measure for the Markov process that is defined by the iterated function system and the given probabilities.

We discuss various aspects that interest us in one of the simplest possible iterated function systems in the following example.

Example 1.2.2. *Let* $X = [0,1]$ *and consider the iterated function system generated by two maps*

$$f_1(x) = \frac{x}{2},$$
$$f_2(x) = \frac{x}{2} + \frac{1}{2}.$$

For any composition f_ω^n *and any* $x, y \in X$ *we find that*

$$|f_\omega^n(x) - f_\omega^n(y) \leq (\frac{1}{2})^n |x-y|$$

goes exponentially fast to zero. The iterated function system is further minimal: for any $x \in X$ *and any open set* $U \subset X$*, one can find a composition* $f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}$ *that maps* x *into* U*:*

$$f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x) \in U.$$

Take f_1 *and* f_2 *independently each iterate with probabilities* p_1 *and* $p_2 = 1 - p_1$*. The corresponding Markov process has a unique stationary measure* m *which has* *a continuous distribution function, see [3, Section 3.4]. Stated differently,* F^+ *has an invariant measure* $\nu^+ \times m$*, that is the unique invariant product measure with* ν^+ *given.*

The skew product system F *has a unique invariant measure* μ *with marginal* ν*. The measure* μ *is given by conditional measures* μ_ω *on the fibers* $\{\omega\} \times X$*. This means*

$$\mu(A) = \int_{\Sigma_k} \mu_\omega(A_\omega)\, d\nu(\omega),$$

where $A_\omega = A \cap (\{\omega\} \times X)$*.*

We have in fact $\mu_\omega = \delta_{X(\omega)}$ *for*

$$X(\omega) = \lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x),$$

where the limit does not depend on the choice of $x \in X$*. One can easily derive an expression for* $X(\omega)$*. For a word* ω *of symbols* $1, 2$ *let* η *be the same word of symbols* $0, 1$*;* $\eta_i = 0$ *if* $\omega_i = 1$ *and* $\eta_i = 1$ *if* $\omega_i = 2$*. Then*

$$X(\omega) = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \eta_{-i}.$$

It follows from these formulas that X *is a continuous map.*

The projection of μ *to* $\Sigma_k^+ \times X$ *by the natural coordinate projection gives the measure* $\nu^+ \times m$*. This means that the distribution of the points* $X(\omega)$ *for fixed* $(\omega_i)_{i=0}^\infty$ *is the stationary measure* m*.*

The above example showed a unique stationary measure. In the context of iterated function systems a unique stationary measure is not immediate. Not even in the context of minimal iterated function systems, such as in the example.

Furstenberg constructed an example of a minimal diffeomorphism f on the two dimensional torus $\mathbb{T}^2$ that has more than one invariant measure, see for instance [24]. This also provides an example of a minimal iterated function system with more than one stationary measure. Simply take the iterated function system generated by $f_1 = f$ and $f_2 = f^{-1}$, where the maps are picked with positive probabilities p_1, p_2. Any invariant measure for f is also an invariant measure for f^{-1} and hence a stationary measure for the corresponding Markov process. We include another example of such nonuniqueness of stationary measures for minimal iterated function systems due to Anthony Quas [29].

Example 1.2.3. *We construct an increasing family of finite words on symbols a and b. Start with*

$$A_0 = a,\ B_0 = b.$$

Now

$$A_n = A_{n-1}^{2^n} B_{n-1} A_{n-1}^{2^n},$$
$$B_n = B_{n-1}^{2^n} A_{n-1} B_{n-1}^{2^n}.$$

The main point about these is that they are constructed so that:

1. *The limiting density of a's in the $A_n s$ is greater than $1/2$ and similarly the limiting density of b's in the B_n is greater than $1/2$.*

2. *Every finite sub-word that appears at any stage appears in all later stages with bounded gaps between occurrences.*

Now define X to be the set of all strings where every finite block appears in some A_n or B_n. By the second item each word appears with bounded gaps. It is not hard to see that this implies that the shift operator on X acts minimal. The shift operator is here defined by $(\sigma\omega)_n = \omega_{n+1}$, for infinite words $\omega = (\omega)_{i=-\infty}^{\infty}$. Take a word A in X whose positive part coincides with A_n for all n. Likewise take a word B in X whose positive part coincides with B_n for all n. One can then obtain an invariant measure by taking empirical measures based on the A word; or you can obtain an invariant measure by taking empirical measures based on the B words. By the first item, these are distinct. In more detail, consider the delta measure δ_A on A and consider averages

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{\sigma^i A}.$$

Limit points (in the weak star topology) are invariant measures. By the first item one can find distinct limit points.

Distinct invariant measures for the shift are also distinct invariant measures for the iterated function system where you randomly shift left or right (that is, under σ or σ^{-1}).

There are situations where minimal iterated function systems do have a unique stationary measure, see [20] and [12]. These examples involve contractive dynamics or negative Lyapunov exponents.

1.3 On notions of attractors

The notion of an attractor is one of the basic concepts of the theory of dynamical systems. There are various notions of attractor.

Consider a dynamical system given by a continuous map f on a complete metric space X. Traditional notions of attractor start with a compact invariant set A with an open neighborhood U so that $f(U) \subset U$ and

$$A = \cap_{n\geq 0} f^n(U).$$

The neighborhood U is called a trapping neighborhood. For A to be an attractor one usually demands in addition the existence of a dense orbit in A or of a point whose forward limit set equals A.

Consider now a map on a compact manifold with a given smooth measure. Generalizing the topological notions of attractor one can consider measure attracting sets: compact sets for which the basin of attraction $\mathcal{B}(A)$ consisting of all points whose forward limit set is contained in A, has positive measure. One adds a condition that a dense orbit exists in A and a minimality condition that no proper compact subset $A' \subset A$ has a basin of positive measure. In this notion of measure theoretic attractor the basin of the attractor need not be open. Measure theoretic attractors can have intermingled basins: there are at least two measure theoretic attractors and every open set intersects each basin in a set of positive measure [6, 18].

For random dynamical systems a reasonable definition of an attractor will be a random set: a stationary set-valued random variable with some notion of convergence. Such approaches to random attractors use the theory of random dynamical systems [1]. During the past decades, the question of existence and properties of a random attractor of a random dynamical system has received considerable attention.

There are a large number of possible ways of defining convergence [2]. The most common and straightforward ones are forward, pullback and weak attraction, see [7]. It is known that these notions of attraction are not equivalent. Fruitful definitions of random attractors in terms of pullback convergence have been proposed by Crauel and Flandoli [8] and Flandoli and Schmalfuß [10]. For pullback convergence one considers orbits starting further and further in the past and ending up in the present. For results related to the existence and structure of attractors using pullback convergence see [27] and [31].

1.4 Thesis outline

In this thesis we discuss iterated function systems generated by finitely many continuous maps $f_1, \ldots, f_k$ on the unit interval $I = [0, 1]$, taken with positive probabilities p_i, $\sum_{i=1}^{k} p_i = 1$, randomly at each iterate. We always assume that these maps fix the boundaries of the interval I and study dynamics of the system under some conditions like the sign of the Lyapunov exponent at the boundaries of the interval. The sign of Lyapunov exponent at a boundary determines whether the boundary is, on average, attracting (negative Lyapunov exponent), neutral (zero), or repelling (positive). We discuss occurrence of intriguing dynamical phenomena such as synchronization, intermingled basins and intermittency, in this context.

Chapter 2. We consider C^2-diffeomorphisms f_1 and f_2 on I that fulfill the following conditions:

1. $f_i(0) = 0$, $f_i(1) = 1$ for $i = 1, 2$;
2. $f_1(x) < x$ for $x \in (0, 1)$;
3. $f_2(x) > x$ for $x \in (0, 1)$.

We deal with the dynamics of step skew product systems $F^+ : \Sigma_2^+ \times I \to \Sigma_2^+ \times I$ of the form

$$F^+(\omega, x) = (\sigma\omega, f_{\omega_0}(x)).$$

where $\Sigma_2^+ = \{1,2\}^{\mathbb{N}}$ and $\sigma : \Sigma_2^+ \to \Sigma_2^+$ is the shift operator; $(\sigma\omega)_i = \omega_{i+1}$.

We review and provide a self-contained discussion of the phenomena: intermingled basins, master-slave synchronization and on-off intermittency, for the chosen class of skew product systems.

Intermingled basins With negative Lyapunov exponents at the boundaries, their basins are intermingled: any open set in $\Sigma_2^+ \times I$ intersects both basins.

Master-slave synchronization We find that for positive Lyapunov exponents at the boundaries the orbits of points in the same fiber synchronize, i.e converge to each other for almost all fibers $\{\omega\} \times I$.

On-off intermittency With zero Lyapunov exponent at a boundary and positive at the other boundary, orbits spend a portion of its iterates with full density near the neutral boundary. With two neutral boundaries, orbits spend a portion of its iterates with full density near the union of the neutral boundaries.

Random walk with drift With one attracting and one repelling or neutral boundary, most orbits approach the attracting boundary.

Chapter 3. In this chapter the iterated function system is given by finitely many logistic maps

$$f_i(x) = \rho_i x(1-x),$$

with $0 < \rho_i \leqslant 4$. The focus is on synchronization and intermittency.

Synchronization We exclude the logistic map with the parameter $\rho = 4$. Also we assume that the Lyapunov exponent at the fixed point 0 is positive so that typical orbits will not converge to 0. We provide sufficient conditions for synchronization, involving negative Lyapunov exponents and minimal dynamics. In a skew product system setting we prove a theorem on synchronization: under identical compositions of logistic maps orbits of different initial conditions converge to each other.

On-off intermittency We analyze a mechanism for intermittency that involves two maps $x \mapsto 2x(1-x)$, for which the critical point is a superstable fixed point, and the map $x \mapsto 4x(1-x)$, for which the critical point is mapped onto 0 in two iterates. We assume that the fixed point at 0 is repelling on average and prove that for typical orbits the set of iterates for which the orbit is near 0, has full density, but orbits do not stay near 0.

Furthermore, we prove the existence of a σ-finite stationary measure for iterated function systems generated by $x \mapsto 2x(1-x)$ and $x \mapsto 4x(1-x)$.

Chapter 4. We aim to describe the dynamics of specific step skew products

$$(\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x))$$

in cases where f_i's are C^2-diffeomorphisms and σ is a transitive subshift of finite type. We provide a classification of dynamics of generic step skew products of such diffeomorphisms on a compact interval which fix the end points of the interval.

A bony graph is a measurable graph and a zero measure set of intervals inside fibers (the bones). In [21] it is shown that bony graphs arise as attractors of generic step skew products of diffeomorphisms over subshifts of finite type. For the diffeomorphisms on a compact interval that fix the endpoints of the interval the situation is different. In addition to bony graphs which have zero standard measure, attractors with positive standard measure as in [14] can occur. This attractors are the closure of an invariant measurable graph and are called thick. We prove that under an assumption of repulsion on average at the end points, both types of graphs, bony and thick, can arise in a single step skew product.

Bibliography

[1] L. Arnold. *Random dynamical systems.* Springer Verlag, 1998.

[2] P. Ashwin and G. Ochs. Convergence to local random attractors. *Dynamical Systems* 18, 139–158, 2003.

[3] R.N. Bhattacharya, M. Majumdar. *Random dynamical systems.* Cambridge University Press, 2007.

[4] M. F. Barnsley. *Fractals Everywhere.* Academic Press, Massachusetts, 1993.

[5] M. F. Barnsley and S. Demko. Iterated function systems and the global construction of fractals. *Proc. R. Soc. Lond. Ser. A*, 399, 243–275, 1985.

[6] C. Bonatti, R. Potrie. Many intermingled basins in dimension 3. To appear in *Israel J. of Math.*

[7] H. Crauel. A uniformly exponential random forward attractor which is not a pull-back attractor. *Archiv der Mathematik* 78, 329–336, 2002.

[8] H. Crauel, F. Flandoli. Attractors for random dynamical systems. *Probability Theory and Related Fields*, 100, 365–393, 1994.

[9] K. Falconer. *Fractal geometry*, Mathematical foundations and applications. Second edition, Wiley,Hoboken, NJ, 2003.

[10] F. Flandoli, B. Schmalfuß. Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise. *Stochastics and Stochastics Reports*, 59, 21–45, 1996.

[11] J. Graczyk, G. Światek Generic hyperbolicity in the logistic family. *Ann. of Math.* 146, 1–52, 1997.

[12] A.J. Homburg. Synchronization in minimal iterated function systems on compact manifolds. preprint, 2016.

[13] J.E. Hutchinson. Fractals and self similarity. *Indiana Univ. Math. Journ.* 30, 713–747, 1981.

[14] Yu. S. Il′yashenko. Thick attractors of step skew products. *Regular and chaotic dynamics*, 15, 328–334, 2010.

[15] M. Iosifescu, S. Grigorescu. *Dependence with complete connections and its applications*, Cambridge University Press, Cambridge, 1990.

[16] P. E. Jorgensen. *Analysis and probability: wavelets, signals, fractals*, Graduate Texts in Mathematics, 234. Springer, New York, 2006.

[17] T. Kaijser. A limit theorem for Markov chains in compact metric spaces with applications to products of random matrices. *Duke Math. J.*, 45, 311–349, 1978.

[18] I. Kan. Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin. *Bull. Amer. Math. Soc.* 31:68–74, 1994.

[19] G. Keller. *Equilibrium states in ergodic theory*, London Mathematical Society Student Texts, 42. Cambridge University Press, Cambridge, 1998.

[20] V. A. Kleptsyn, M. B. Nal′skiĭ. Contraction of orbits in random dynamical systems on the circle. *Funct. Anal. Appl.* 38, 267–282, 2004.

[21] V. A. Kleptsyn, D. Volk. Physical measures for nonlinear random walks on interval. *Mosc. Math. J.*, 14, 339–365, 2014.

[22] M. Lyubich. Almost every real quadratic map is either regular or stochastic. *Annals of Math.* 156, 1–78, 2002.

[23] B. Mandelbrot. *Fractals: form, chance, and dimension.* W. H. Freeman and Co., San Francisco, Calif., 1977.

[24] R. Mañé. *Ergodic theory and differentiable dynamics*, Springer-Verlag, 1987.

[25] W. de Melo, S. van Strien. *One-dimensional dynamics*, Springer-Verlag, 1993.

[26] C. Mira. *Chaotic Dynamics. From the one-dimensional endomorphism to the two-dimensional diffeomorphism.* , World Scientific, Singapore, 1987.

[27] G. Ochs. *Weak random attractors*, Report 449, Institut für Dynamische Systeme, Bremen 1999.

[28] Y. B. Pesin. *Dimension theory in dynamical systems. Contemporary views and applications.* Chicago Lectures in Mathematics. University of Chicago Press, 1997.

[29] A. Quas. Personal communication, 2015.

[30] G. O. Roberts, J. S. Rosenthal. General state space Markov chains and MCMC algorithms. *Probab. Surv.*, 1:20–71, 2004.

[31] K. R. Schenk-Hoppé. Random attractors, general properties, existence and applications to stochastic bifurcation theory. *Discrete Contin. Dyn. Syst.* 4, 99–130, 1998.

[32] O. Stenflo. Uniqueness of invariant measures for place-dependent random iterations of functions. *IMA Vol. Math. Appl.*, 132, 13–32, 2002.

[33] E. R. Vrscay. Iterated function systems: theory, applications and the inverse problem. In *Proceedings of the NATO Advanced Study Institute on Fractal Geometry held in Montreal, July 1989.* Kluwer Academic Pulishers, 1990.

[34] R. F. Williams. Composition of contractions. *Bol. Soc. Brasil. Mat.*, 2, 55–59, 1971.

Chapter 2

Random interval diffeomorphisms

This chapter is the article :
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Abstract
We consider a class of step skew product systems of interval diffeomorphisms over shift operators, as a means to study random compositions of interval diffeomorphisms. The class is chosen to present in a simplified setting intriguing phenomena of intermingled basins, master-slave synchronization and on-off intermittency. We provide a self-contained discussion of these phenomena.

2.1 Introduction

We deal with the dynamics of specific step skew product systems $F^+ : \Sigma_2^+ \times I \to \Sigma_2^+ \times I$, where $\Sigma_2^+ = \{1,2\}^{\mathbb{N}}$ and $I = [0,1]$, of the form

$$F^+(\omega, x) = (\sigma\omega, f_{\omega_0}(x)).$$

Here $\sigma : \Sigma_2^+ \to \Sigma_2^+$ is the shift operator; $(\sigma\omega)_i = \omega_{i+1}$ for $\omega = (\omega_i)_0^\infty$ and f_1, f_2 are C^2-diffeomorphisms on I that fulfill the following conditions:

1. $f_i(0) = 0$, $f_i(1) = 1$ for $i = 1, 2$;
2. $f_1(x) < x$ for $x \in (0, 1)$;
3. $f_2(x) > x$ for $x \in (0, 1)$.

We review and present a self-contained study of the dynamics of such skew product systems, characterizing the different possible dynamics. This may seem a restrictive setup, but these systems exhibit a wealth of dynamical behavior that serves as models for dynamics in more general systems.

These step skew product systems provide a setting to study all possible compositions of the two maps f_1, f_2 in a single framework. Indeed, for initial conditions $(\omega, x) \in \Sigma_2^+ \times I$, the coordinate in I iterates as

$$x,\ f_{\omega_0}(x),\ f_{\omega_1} \circ f_{\omega_0}(x),\ f_{\omega_2} \circ f_{\omega_1} \circ f_{\omega_0}(x),\ ... \tag{2.1.1}$$

The maps f_1, f_2 simply move points to either smaller or larger values. We will pick the diffeomorphisms f_1 and f_2 randomly, independently at each iterate, with positive probabilities p_1 and $p_2 = 1 - p_1$. This corresponds to taking a Bernoulli measure on Σ_2^+ from which we pick ω. The obtained random compositions (2.1.1) thus form a (nonhomogeneous) random walk on the interval.

The dynamics of the step skew product system depends on the Lyapunov exponents at the boundaries $\Sigma_2^+ \times \{0\}$ and $\Sigma_2^+ \times \{1\}$. We list the possibilities, which will be worked out in subsequent sections below.

Intermingled basins With negative Lyapunov exponents at the boundaries, these boundaries are attracting. Their basins are intermingled: any open set in $\Sigma_2^+ \times I$ intersects both basins.

Master-slave synchronization With positive Lyapunov exponents at the boundaries, the boundaries are repelling. We find that for almost all fibers $\{\omega\} \times I$, orbits of points in the same fiber converge to each other, i.e. synchronize.

On-off intermittency A zero Lyapunov exponent at a boundary makes that boundary neutral. With the other boundary repelling, a typical time series has long laminary phases where the orbit is close to the "off" state (the neutral boundary) and has bursts where the orbit is in the "on" state, i.e. away from the neutral boundary. Orbits spend a portion of its iterates with full density near the neutral boundary.

With two neutral boundaries, orbits spend a portion of its iterates with full density near the union of the neutral boundaries.

Random walk with drift With one attracting and one repelling or neutral boundary, most orbits approach the attracting boundary.

Thus we find the most elementary case of the more widespread phenomenon of intermingled basins [1, 25], or on-off intermittency [38, 21], or master-slave synchronization [36, 40].

The setup chosen in this paper is a starting point for research in random dynamics, see e.g. [4], and nonhyperbolic dynamics, see e.g. [10], and has relations to nonautonomous dynamics, see e.g. [30]. The following directions for generalizations give an idea of the many possibilities. We will not give details, but refer to [4, 10, 30] for more. One may consider other measures than Bernoulli measures to pick random compositions of the interval maps. A natural generalization is also to let the diffeomorphisms on I depend on ω more generally than through ω_0 alone;

$$(\omega, x) \mapsto (\sigma\omega, f_\omega(x)).$$

One can further consider parameters ω from other spaces than symbol spaces, with other dynamics than generated by the shift operator. One may then also generalize the skew product structure to maps on fiber bundles, and study perturbations that destroy the skew product structure. A heuristic principle going back to [19] states that phenomena in random dynamics on compact manifolds may also occur for diffeomorphisms of manifolds of higher dimensions.

This paper is organized as follows. We start with a section that contains definitions. The next sections form the heart of the paper, describing possible dynamics for the considered class of step skew product systems. An important role in the study of skew product systems is by invariant measures. A basic result gives the connection between invariant measures for skew product systems and their natural extensions. In the appendix this is worked out in the simple context of step skew product systems over one-sided and two-sided shifts.

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2.2 Step skew product systems

This section serves to present the setup of this paper and to collect necessary definitions. A skew product system is a dynamical system generated by a map $F : Y \times X \to Y \times X$ of the form

$$F(y, x) = (g(y), f(y, x)); \tag{2.2.1}$$

if one sees X as the state space of interest, one has dynamics of the x variable that is governed by the map f which depends on the variable y that changes through g. The space Y is the base space, the sets $\{y\} \times X$ are fibers.

We have an interest in skew product systems over full shifts. Write Ω for the finite set of symbols $\{1, \ldots, N\}$. Let $\Sigma_N = \Omega^{\mathbb{Z}}$ be the set of bilateral sequences $\omega = (\omega_n)_{-\infty}^{\infty}$ composed of symbols in Ω. Let $\sigma : \Sigma_N \to \Sigma_N$ be the shift operator; the map σ shifts every sequence $\omega \in \Sigma_N$ one step to the left, $(\sigma\omega)_i = \omega_{i+1}$. We can also consider the shift operator σ acting on the one-sided symbol space Σ_N^+, i.e. the space of sequences $\omega = (\omega_n)_0^{\infty}$ composed of symbols in Ω. The spaces Σ_N and Σ_N^+ are endowed with the product topology. This topology is generated by cylinders like $C_{\omega_1,\ldots,\omega_n}^{k_1,\ldots,k_n}$ for Σ_N,

$$C_{\omega_1,\ldots,\omega_n}^{k_1,\ldots,k_n} = \{\omega' \in \Sigma_N \ ; \ \omega'_{k_i} = \omega_{k_i}, \ \forall i = 1, \ldots, n\}.$$

As it will not lead to confusion, we use the same notation for cylinders in Σ_N^+.

Now let M be a compact manifold, or compact manifold with boundary, and for $\omega \in \Sigma_N$, let $f_\omega : M \to M$ be diffeomorphisms depending continuously on ω.

Consider skew product systems $F : \Sigma_N \times M \to \Sigma_N \times M$;

$$F(\omega, x) = (\sigma\omega, f_\omega(x)).$$

Definition 2.2.1. *A skew product system $F : \Sigma_N \times M \to \Sigma_N \times M$ is a step skew product system if it is of the form*

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x)),$$

i.e. the fiber maps depend on ω_0 alone.

We denote iterates of a skew product system $F(\omega, x) = (\sigma\omega, f_\omega(x))$ as

$$F^n(\omega, x) = (\sigma^n\omega, f^n_\omega(x)).$$

Here, for $n \geq 1$,

$$f^n_\omega(x) = f_{\sigma^{n-1}\omega} \circ \cdots \circ f_\omega(x).$$

For a step skew product system this becomes

$$f^n_\omega(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x).$$

Observe that, if $-n < 0$,

$$f^{-n}_\omega(x) = \left(f^n_{\sigma^{-n}\omega}\right)^{-1}.$$

We also consider (step) skew products over the shift on one-sided symbol sequences. We write F^+ for the skew product system

$$F^+(\omega, x) = (\sigma\omega, f_\omega(x))$$

on $\Sigma^+_N \times M$. Recall that a natural extension of a continuous map is the smallest invertible extension, up to topological semi-conjugacy. The skew product system F on $\Sigma_N \times M$ is the natural extension of F^+ on $\Sigma^+_N \times M$.

Definition 2.2.2. *Let $\mathbb{F}$ be a family of diffeomorphisms on M. The iterated function system IFS$(\mathbb{F})$ is the action of the semigroup generated by $\mathbb{F}$.*

So a collection of diffeomorphisms f_i, $1 \leq i \leq N$, generates an iterated function system. And an iterated function system IFS $(\{f_1, \ldots, f_N\})$ on M corresponds to a step skew product system $F^+(\omega, x) = (\sigma\omega, f_{\omega_0}(x))$ on $\Sigma^+_N \times M$. Given an iterated function system IFS $(\mathbb{F})$, a sequence $\{x_n : n \in \mathbb{N}\}$ is called a branch of an orbit of IFS $(\mathbb{F})$ if for each $n \in \mathbb{N}$ there is $f_n \in \mathbb{F}$ such that $x_{n+1} = f_n(x_n)$. We say that IFS $(\mathbb{F})$ is minimal if every orbit has a branch which is dense in M.

The appendix collects definitions and basic results on stationary and invariant measures in the context of step skew product systems over shifts. We will make use of the material from the appendix in the following sections.

2.2.1 Interval fibers

Focus of this paper is the following class of step skew products of diffeomorphisms on $I = [0,1]$ over the full shift on two symbols $\{1,2\}$, earlier presented in the introduction.

Definition 2.2.3. *Let $\mathcal{S}$ be the set of step skew product systems $F^+ : \Sigma_2^+ \times I \to \Sigma_2^+ \times I$ with*

$$F^+(\omega, x) = (\sigma\omega, f_{\omega_0}(x)),$$

where f_1, f_2 are C^2 diffeomorphisms that fulfill the following conditions:

1. *$f_i(0) = 0$, $f_i(1) = 1$ for $i = 1, 2$;*
2. *$f_1(x) < x$ for $x \in (0,1)$;*
3. *$f_2(x) > x$ for $x \in (0,1)$.*

So f_1 moves points in $(0,1)$ to the left, whereas f_2 moves points in $(0,1)$ to the right. On Σ_2^+ we take Bernoulli measure ν^+ where the symbols $1,2$ have probability p_1, p_2, see Appendix 2.A.

The (fiber) Lyapunov exponent of F^+ at a point $(\omega, x) \in \Sigma_2^+ \times I$ is

$$\lim_{n\to\infty} \frac{1}{n} \ln \Big(f'_{\omega_{n-1}} \left(f_\omega^{n-1}(x) \right) \cdots f'_{\omega_0}(x) \Big) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left(f'_{\sigma^i\omega}(f_\omega^i(x)) \right),$$

in case the limit exists. Since $x = 0, 1$ are fixed points of f_i, $i = 1, 2$, by Birkhoff's ergodic theorem, we obtain for $x = 0, 1$ that

$$L(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left(f'_{\sigma^i\omega}(x) \right) = \int_{\Sigma_2^+} \ln \left(f'_\omega(x) \right) \, d\nu^+(\omega) = \sum_{i=1}^{2} p_i \ln \left(f'_i(x) \right)$$

for ν^+-almost all $\omega \in \Sigma_2^+$.

Definition 2.2.4. *The standard measure s on $\Sigma_2^+ \times I$ is the product of Bernoulli measure ν^+ and Lebesgue measure on I.*

A specific example of a step skew product system from $\mathcal{S}$ comes from the symmetric random walk. The symmetric random walk is given by translations

$$k_1(x) = x - 1,$$
$$k_2(x) = x + 1$$

on the real line, where both maps are chosen randomly with probability $1/2$. Now conjugate the symmetric random walk to maps on I as follows. Consider the coordinate change given by the diffeomorphism $h : \mathbb{R} \to (0,1)$,

$$h(x) = \frac{e^x}{1 + e^x}$$

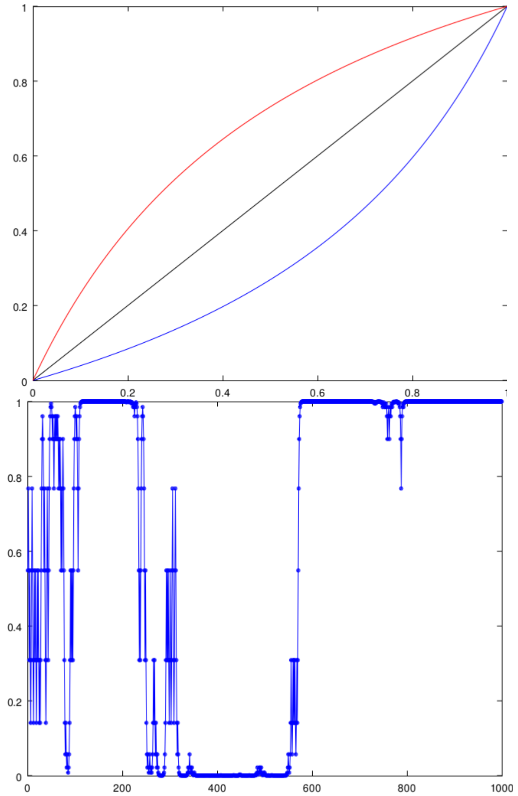

Figure 2.1: *The left frame depicts the graphs of* g_1, g_2*, the diffeomorphisms on* I *that are conjugate to the maps* $y \mapsto y \pm 1$ *that generate the symmetric random walk. The right frame shows a time series of the iterated function system generated by* g_1, g_2*, both picked with probability* $1/2$*.*

(note that $h^{-1}(x) = \ln(x/(1-x))$). Define the step skew product system on $\Sigma_2 \times I$ generated by the fiber diffeomorphisms $g_i = h \circ k_i \circ h^{-1}$ with $g_i(0) = 0$, $g_i(1) = 1$, $i = 1, 2$. We have

$$g_1(x) = \frac{\frac{1}{e}x}{1 + (\frac{1}{e} - 1)x}, \tag{2.2.2}$$

$$g_2(x) = \frac{ex}{1 + (e-1)x}, \tag{2.2.3}$$

see Figure 2.1. We will also refer to the step skew product system generated by g_1, g_2 as the symmetric random walk.

Observe $g_1'(0) = \frac{1}{e}$, $g_2'(0) = e$, $g_1'(1) = e$, $g_2'(1) = \frac{1}{e}$, so that the symmetric random walk has zero Lyapunov exponents at the boundaries $\Sigma_2^+ \times \{0\}$ and $\Sigma_2^+ \times \{1\}$. Perturbations of g_1, g_2 that preserve the boundary points $0, 1$ lead to diffeomorphisms f_1, f_2 with various signs of Lyapunov exponents at the boundaries: all cases that are treated in the following sections also occur as small perturbations from the symmetric random walk.

The text book [12] contains a discussion of recurrence properties of random walks on the line with i.i.d. steps. In the same vein one can ask for the iterated function system IFS $(\{f_1, f_2\})$ to be minimal on $(0, 1)$. The proof of [22, Lemma 3] gives the following result.

Proposition 2.2.5. *Assume that* $\lambda = f_1'(0) < 1$, $\mu = f_2'(0) > 1$. *Assume further that either*

$$\ln(\lambda)/\ln(\mu) \notin \mathbb{Q},$$

or

$$\frac{f_1''(0)}{\lambda^2 - \lambda} \neq \frac{f_2''(0)}{\mu^2 - \mu}.$$

Then the iterated function system generated by f_1, f_2 *is minimal on* $(0, 1)$. *Such minimality is also implied by analogous conditions at the end point* 1.

Proof. For the proof we refer to [22]. We add some comments to clarify the conditions. Il'yashenko [22, Lemma 3] considers, for $x, y \in (0, 1)$, compositions $f_2^l \circ f_1^k(x)$ that converge to y for suitable $k, l \to \infty$. Note that this property implies minimality. His analysis uses linearizing coordinates $h \circ f_1 \circ h^{-1}(x) = \lambda x$ with $x \in [0, s]$ for an $s < 1$. Here h is a local diffeomorphism. The two cases where $\ln(\lambda), \ln(\mu)$ are rationally dependent or not, are distinguished. In case $\ln(\lambda), \ln(\mu)$ are rationally dependent, the argument works if the second order derivative of $h \circ f_2 \circ h^{-1}$ at 0 is not zero. An explicit calculation shows that this gives the condition in the proposition. □

Obviously, the iterated function system generated by g_1 and g_2, where $g_2 = g_1^{-1}$, is not minimal.

2.3 Intermingled basins

Kan [25] describes an example of a skew product system on $\mathbb{T}\times I$, over an expanding circle map in the base, where the boundary components $\mathbb{T} \times \{0\}$ and $\mathbb{T} \times \{1\}$ are attractors so that both basins intersect each open set. We will describe his results in the elementary setting of step skew product systems.

The following result describes intermingled basins for step skew product systems $F^+ \in \mathcal{S}$.

Theorem 2.3.1. *Let $F^+ \in \mathcal{S}$ and assume $L(0) < 0$ and $L(1) < 0$. The sets $\Sigma_2^+ \times \{0\}$ and $\Sigma_2^+ \times \{1\}$ attract sets of positive standard measure. Both their basins lie dense in $\Sigma_2^+ \times I$. The union of the basins has full standard measure.*

Let $F : \Sigma_2 \times I \to \Sigma_2 \times I$ denote the natural extension of F^+. There is an invariant measurable graph $\xi : \Sigma_2 \to I$ that separates the basins: for ν-almost all ω,

$$\lim_{n\to\infty} f_\omega^n(x) = \begin{cases} 0, & \text{if } x < \xi(\omega), \\ 1, & \text{if } x > \xi(\omega). \end{cases}$$

We note that $\Sigma_2^+ \times \{0\}$ and $\Sigma_2^+ \times \{1\}$ are attractors in Milnor's sense [34]. The values $\xi(\omega)$ depend only on the present and future coefficients $(\omega_i)_0^\infty$. Before starting the actual proof, we provide a simple argument showing positive standard measure of the basins of $\Sigma_2^+ \times \{0\}$ and $\Sigma_2^+ \times \{1\}$.

Lemma 2.3.2. *Let $F^+ \in \mathcal{S}$ and assume $L(0) < 0$. Let*

$$r(\omega) = \sup\{x \in I \mid \lim_{n\to\infty} f_\omega^n(x) = 0\}.$$

Then $r(\omega) > 0$ for ν^+-almost all $\omega \in \Sigma_2^+$.

Proof. The argument follows [25, Lemma 2.2] or [11, Lemma A.1]. For any $\varepsilon > 0$ there exists $\delta > 0$ so that

$$f_i'(x) \leq f_i'(0) + \varepsilon$$

if $x < \delta$, for both $i = 1, 2$. Write $a_i = \ln(f_i'(0) + \varepsilon)$. Recall that $L(0) = p_1 \ln(f_1'(0)) + p_2 \ln(f_2'(0))$ is negative by assumption. By Birkhoff's ergodic theorem applied to the function $\omega \mapsto \ln(f_\omega'(0) + \varepsilon)$, for ν^+-almost all ω,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} a_{\omega_i} = p_1 a_1 + p_2 a_2,$$

which is negative if ε is small enough. So, for ν^+-almost all ω, $\sum_{i=0}^{n-1} a_{\omega_i}$ goes to $-\infty$ as $n \to \infty$ and

$$A(\omega) = \max\{0, \max_{n\geq 1} \sum_{i=0}^{n-1} a_{\omega_i}\}$$

exists. Take $x_0 < \delta e^{-A(\omega)} \leq \delta$. Then $x_n = f_\omega^n(x_0)$ satisfies

$$x_n < e^{\sum_{i=0}^{n-1} a_{\omega_i}} e^{-A(\omega)} \delta \leq \delta$$

for all $n \geq 0$ and in fact $\lim_{n\to\infty} x_n = 0$. This proves the lemma. $\square$

Since the function r is positive almost everywhere, the basin of $\Sigma_2^+ \times \{0\}$ has positive standard measure. The same holds for the basin of $\Sigma_2^+ \times \{1\}$. It is easily seen that any open set in $\Sigma_2^+ \times I$ intersects both basins; forward iterations must accumulate onto both $\Sigma_2^+ \times \{0\}$ and $\Sigma_2^+ \times \{1\}$ using that the shift operator is an expansion and 0 and 1 occur as attracting fixed points for f_1 and f_2 respectively.

Proof of Theorem 2.3.1. We prove the theorem by considering the inverse diffeomorphisms, i.e. a step skew product with positive Lyapunov exponents along $\Sigma_2^+ \times \{0\}$ and $\Sigma_2^+ \times \{1\}$. For the duration of this proof, we consider $F^+ \in \mathcal{S}$ with $L(0) > 0$ and $L(1) > 0$. The following lemmas deal with this. The theorem will follow by linking the derived statements on the natural extension F of F^+ and the statements we wish to prove for its inverse.

We write $\mathcal{P}_I$ for the space of probability measures on I equipped with the weak star topology. As explained in Appendix 2.A, a stationary measure is a fixed point of $\mathcal{T} : \mathcal{P}_I \to \mathcal{P}_I$ given by

$$\mathcal{T}m = p_1 f_1 m + p_2 f_2 m,$$

where $f_i m$ is the push forward measure $f_i m(A) = m(f_i^{-1}(A))$.

Lemma 2.3.3. *Let $F^+ \in \mathcal{S}$ and assume $L(0) > 0$ and $L(1) > 0$. Then there exists an ergodic stationary measure m with $m(\{0\} \cup \{1\}) = 0$.*

Proof. For small $0 < \alpha < 1$, $q > 0$, and positive c, define

$$\mathcal{N}_c = \{m \in \mathcal{P}_I \ ; \ \forall\ 0 \le x \le q,\ m\big([0,x)\big) \le cx^\alpha \text{ and } \ m\big((1-x,1]\big) \le cx^\alpha\}.$$

The conditions exclude stationary measures supported on the end points 0 or 1. Note that $\mathcal{N}_c$ depends on α and q; but we do not include this dependence in the notation. We first show that there exist $c > 0$ and $\alpha, q > 0$ close to 0 such that $\mathcal{T}(\mathcal{N}_c) \subset \mathcal{N}_c$.

Write $\rho_i = f_i'(0)$. We claim that there is a small $\alpha > 0$ such that the assumption $L(0) > 0$ implies $\sum_{i=1}^2 p_i \rho_i^{-\alpha} < 1$. Namely, since $\lim\limits_{\alpha \to 0} \frac{1-\rho_i^{-\alpha}}{\alpha} = \ln \rho_i$ for $1 \le i \le 2$, $\sum_{i=1}^2 p_i \ln \rho_i > 0$ implies that for sufficiently small $\alpha > 0$,

$$\sum_{i=1}^{2} p_i \frac{1-\rho_i^{-\alpha}}{\alpha} > 0.$$

Multiplying by α we get

$$\sum_{i=1}^{2} p_i - \sum_{i=1}^{2} p_i \rho_i^{-\alpha} > 0,$$

which implies $\sum_{i=1}^2 p_i \rho_i^{-\alpha} < 1$, because $\sum_{i=1}^2 p_i = 1$.

Thus, there exists a small $\delta > 0$ so that

$$\sum_{i=1}^{2} \frac{p_i}{(\rho_i - \delta)^\alpha} < 1. \tag{2.3.1}$$

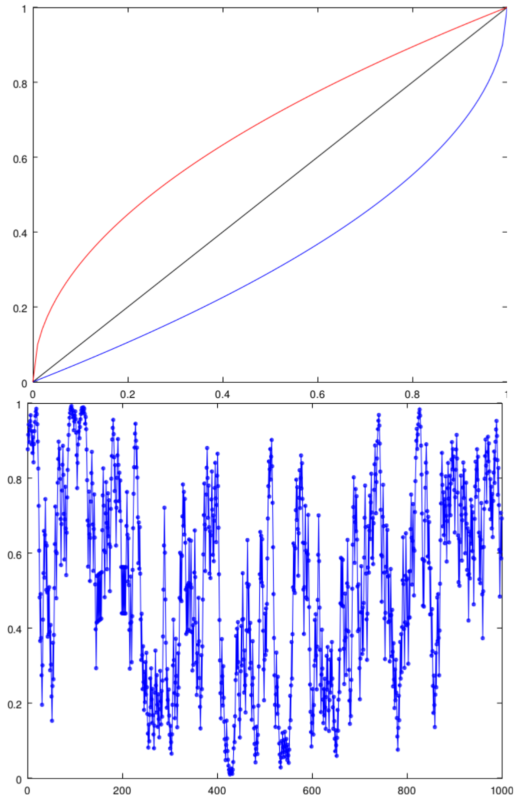

Figure 2.2: *With* $r = 1/2$*, the diffeomorphisms* $f_1(x) = x - rx(1-x)$ *and* $f_2(x) = x + rx(1-x)$ *(picked with probabilities* $1/2$*) give negative Lyapunov exponents at the end points* $0, 1$*. Depicted, in the left frame, are the graphs of the inverse diffeomorphisms* $f_1^{-1}(x) = \frac{1-r-\sqrt{(1-r)^2+4rx}}{-2r}$ *and* $f_2^{-1}(x) = \frac{1+r-\sqrt{(1+r)^2-4rx}}{2r}$*. The inverse maps give positive Lyapunov exponents at the end points. The right frame shows a time series for the iterated function system generated by* f_1^{-1} *and* f_2^{-1}*.*

Moreover, for such $\delta > 0$ we are able to choose a sufficiently small $q = q(\delta) > 0$ so that

$$f_i^{-1}(x) \leq \frac{x}{\rho_i - \delta},\ \forall\, 0 \leq x \leq q. \tag{2.3.2}$$

Take c with

$$cq^\alpha > 1.$$

Note that this implies that in the definition of $\mathcal{N}_c$, $m\big([0,x)\big) \leq cx^\alpha$ and $m\big((1-x,1]\big) \leq cx^\alpha$ for any $0 \leq x \leq 1$, and not just for $0 \leq x \leq q$. Take a measure $m \in \mathcal{N}_c$. To prove $\mathcal{T}m \in \mathcal{N}_c$, we must show that for $x \leq q$, $\mathcal{T}m\big([0,x)\big) \leq cx^\alpha$. Knowing that $m\big([0,x)\big) \leq cx^\alpha$ and applying (2.3.1), (2.3.2) we obtain the following estimates:

$$\begin{aligned}\mathcal{T}m\big([0,x)\big) &= \sum_{i=1}^{2} p_i f_i m\big([0,x)\big) = \sum_{i=1}^{2} p_i m\left(f_i^{-1}[0,x)\right) \leq \sum_{i=1}^{2} p_i m\left([0, \frac{x}{\rho_i - \delta})\right) \\ &\leq \sum_{i=1}^{2} p_i c \left(\frac{x}{\rho_i - \delta}\right)^\alpha = c\left(\sum_{i=1}^{2} \frac{p_i}{(\rho_i - \delta)^\alpha}\right) x^\alpha \leq cx^\alpha. \end{aligned} \tag{2.3.3}$$

Estimates near the right boundary point are treated in the same manner.

By the Krylov-Bogolyubov averaging method, for a measure $m \in \mathcal{N}_c$ there is a subsequence of $\{\frac{1}{n}\sum_{r=0}^{n-1} \mathcal{T}^r m\}_{n\in\mathbb{N}}$ that is convergent to a probability measure $\hat{m} \in \mathcal{N}_c$ such that $\mathcal{T}\hat{m} = \hat{m}$.

The following additional reasoning shows that there is an ergodic stationary measure in $\mathcal{N}_c$. The set of stationary measures $\mathcal{M}_I$ is a convex compact subset of $\mathcal{P}_I$. The ergodic stationary measures are the extreme points of it. Note that $\mathcal{N}_c \cap \mathcal{M}_I$ is a convex compact subset of $\mathcal{M}_I$, which is itself also convex and compact. We claim that the extreme points of $\mathcal{N}_c \cap \mathcal{M}_I$ are also extreme points of $\mathcal{M}_I$. Suppose by contradiction that there are $n_1, n_2 \in \mathcal{M}_I \setminus (\mathcal{N}_c \cap \mathcal{M}_I)$ and the convex combination $m = sn_1 + (1-s)n_2 \in \mathcal{N}_c \cap \mathcal{M}_I$. In this case, for $0 \leq x \leq q$, $n_1([0,x)) \leq (c/s)x^\alpha$ and $n_1((1-x,1]) \leq (c/s)x^\alpha$ and similar estimates for n_2. That is, $x \mapsto n_i([0,x))/x^\alpha$ and $x \mapsto n_i((1-x,1])/x^\alpha$ are bounded. As $\mathcal{T}m = m$, we have by (2.3.1), (2.3.3) that $m \in \mathcal{N}_{\tilde{c}}$ for some $\tilde{c} < c$. It follows that $tn_1 + (1-t)n_2 \in \mathcal{N}_c \cap \mathcal{M}_I$ for t close to s. So s is an interior point of the set of values t for which $tn_1 + (1-t)n_2 \in \mathcal{N}_c \cap \mathcal{M}_I$. Since $\mathcal{N}_c \cap \mathcal{M}_I$ is closed it follows that $n_i \in \mathcal{N}_c \cap \mathcal{M}_I$ and the claim is proved. Since the extreme points of $\mathcal{M}_I$ are ergodic stationary measures, we conclude that the extreme points of $\mathcal{N}_c \cap \mathcal{M}_I$ are ergodic stationary measures. Since the Krein-Milman theorem the set of extreme points of $\mathcal{N}_c \cap \mathcal{M}_I$ is nonempty, there are ergodic stationary measures in $\mathcal{N}_c$. $\square$

A stationary measure m gives an invariant measure μ_m for the step skew product system F, as explained in Appendix 2.A. Its conditional measures on fibers $\{\omega\} \times I$ are denoted by $\mu_{m,\omega}$.

Lemma 2.3.4. *For every ergodic stationary probability measure m, the conditional measure $\mu_{m,\omega}$ of μ_m is a δ-measure for ν-almost every $\omega \in \Sigma_2$.*

Proof. We follow [4, Theorem 1.8.4]. Consider a μ_m and its conditional measures $\mu_{m,\omega}$. Let $X_m(\omega)$ be the smallest median of $\mu_{m,\omega}$, i.e. the infimum of all points x for which

$$\mu_{m,\omega}([0,x]) \geq \frac{1}{2} \text{ and } \mu_{m,\omega}([x,1]) \geq \frac{1}{2}.$$

The set of medians of μ_m is a compact interval and $X_m : \Sigma_2 \to I$ is measurable. Define $J_m^-(\omega) = [0, X_m(\omega)]$ for which by definition $\mu_{m,\omega}(J_m^-(\omega)) \geq \frac{1}{2}$. The set $J_m^-(\omega)$ is invariant: since f_1 and f_2 are increasing, for every $x_1 < x_2$ and ω we have $f_\omega(x_1) < f_\omega(x_2)$. This implies that x is a median of $\mu_{m,\omega}$ if and only if $f_\omega(x)$ is a median of $f_\omega \mu_{m,\omega}$. By invariance of μ_m for F we have

$$f_\omega \mu_{m,\omega} = \mu_{m,\sigma\omega}.$$

Hence, $X_m(\sigma\omega) = f_\omega(X_m(\omega))$ which implies $J_m^-(\sigma\omega) = f_\omega(J_m^-(\omega))$.

Because μ_m is ergodic and $J_m^-(\omega)$ is invariant, $\mu_{m,\omega}(J_m^-(\omega)) = 1$, ν-almost surely. Applying the same argument to $J_m^+(\omega) = [X_m(\omega), 1]$, we obtain

$$\mu_{m,\omega}(\{X_m(\omega)\}) = 1$$

for $\{X_m(\omega)\} = J_m^-(\omega) \cap J_m^+(\omega)$. Thus $\mu_{m,\omega} = \delta_{X_m(\omega)}$ for ν-almost every $\omega \in \Sigma_2$. □

The following lemma shows that the set of stationary measures is the triangle consisting of convex combinations of δ_0, δ_1 and one other ergodic stationary measure m.

Lemma 2.3.5. *There is a unique stationary measure m with $m(\{0\} \cup \{1\}) = 0$.*

Proof. Suppose there are two different such stationary measures m_1, m_2. We may take m_1, m_2 to be ergodic stationary measures. This corresponds to two ergodic invariant measures $\mu_{m_1} \neq \mu_{m_2}$ for F. By Proposition 2.A.4 and Lemma 2.3.4 there are measurable functions $X_{m_i} : \Sigma_2 \to I$ and sets $D_i \subset \Sigma_2$ with $\nu(D_i) = 1$, for $i = 1, 2$, such that

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega} m_i = \delta_{X_{m_i}(\omega)}$$

for every $\omega \in D_i$. From $\nu(D_i) = 1$ we have $\nu(D_1 \cap D_2) = 1$. Since μ_{m_1}, μ_{m_2} are mutually singular we have that for a generic $\bar{\omega} \in D_1 \cap D_2$, $X_{m_1}(\bar{\omega}) \neq X_{m_2}(\bar{\omega})$. Without loss of generality suppose that $X_{m_1}(\bar{\omega}) < X_{m_2}(\bar{\omega})$.

Observe that the supports of m_1 and m_2 are invariant:

$$\operatorname{supp}(m_i) = f_1(\operatorname{supp}(m_i)) \cup f_2(\operatorname{supp}(m_i))$$

for $i = 1, 2$. The convex hulls of the supports of m_1 and m_2 therefore both equal I. We can find generic points $(\bar{\omega}, x_1)$ and $(\bar{\omega}, x_2)$ for m_1 and m_2 such that $x_1 > x_2$. Because

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\bar{\omega}}(x_i) = X_{m_i}(\bar{\omega})$$

and f_1, f_2 are both strictly increasing, we conclude that $X_{m_2}(\bar{\omega}) < X_{m_1}(\bar{\omega})$, contradicting our assumption. Thus, $\mu_{m_1} = \mu_{m_2}$. □

By Proposition 2.A.4, Lemmas 2.3.4 and 2.3.5, there exists a measurable function $\xi : \Sigma_2 \to I$ such that

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega} m = \delta_{\xi(\omega)},$$

for ν-almost all ω, where m is the stationary measure with $m(\{0\} \cup \{1\}) = 0$. As the convex hull of the support of m equals $\mathbb{I}$ and f_1, f_2 are increasing, this implies

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x) = \xi(\omega)$$

for every $x \in (0, 1)$. Again since the diffeomorphisms f_1, f_2 are increasing, for the inverse diffeomorphisms $(f^n_{\sigma^{-n}\omega})^{-1} = f^{-1}_{\omega_{-n}} \circ \cdots \circ f^{-1}_{\omega_{-1}}$ this yields

$$\lim_{n\to\infty} (f^n_{\sigma^{-n}\omega})^{-1}(y) = 1$$

if $y > \xi(\omega)$ and

$$\lim_{n\to\infty} (f^n_{\sigma^{-n}\omega})^{-1}(y) = 0$$

if $y < \xi(\omega)$. Note that this also shows that for the inverse diffeomorphisms, the union of the basins of attraction of $\Sigma_2 \times \{0\}$ and $\Sigma_2 \times \{1\}$ has full standard measure. $\square$

Let us give some pointers to further research literature: a discussion of a step skew product system over the full shift on two symbols, with piecewise linear fiber maps is in [2]. Several articles discuss extensions to skew product systems that are not step skew product systems. We refer the reader in particular to [28, 24] and [10, Section 11.1]. Further studies that quantify the phenonomenon are [35, 26]. See [22, 23, 18] for related work on so-called thick attractors (attractors of positive standard measure).

2.4 Master-slave synchronization

The proof of Theorem 2.3.1 relies on an analysis of step skew product systems with positive Lyapunov exponents at the boundaries. The following result further discusses such step skew product systems. It describes a synchronization phenomenon that is illustrated in the right frame of Figure 2.3. We see an example of master-slave synchronization, which refers to synchronization caused by external forcing. It is explained by a single attracting invariant graph for the skew product system [40]. From a slightly different perspective one can also view this as synchronization by noise, where common noise synchronizes orbits with different initial conditions.

In a general context of skew product systems $F(y, x) = (g(y), f(y, x))$ on a product $Y \times X$ of metric spaces Y, X, as in (2.2.1), master-slave synchronization is given by the following picture. If $\{(y, \xi(y)) \mid y \in Y\}$ is a globally attracting graph,

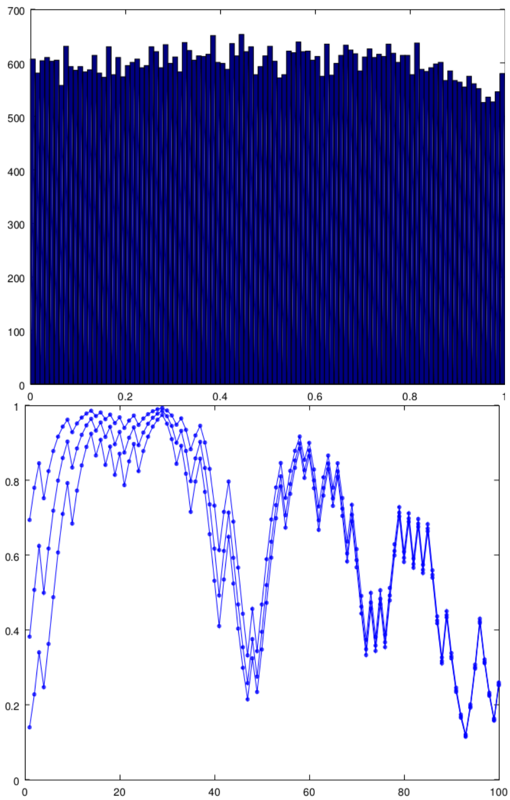

Figure 2.3: *The left frame shows a numerically computed histogram for a time series of the iterated function systems generated by the same diffeomorphisms* f_1^{-1} *and* f_2^{-1} *used in Figure 2.2. The right frame indicates asymptotic convergence of orbits within fibers: it depicts time series for three different initial conditions in* I *with the same* ω*.*

then the orbits $F^n(y, x_1)$ and $F^n(y, x_2)$ converge to each other. In particular if one observes the dynamics of the x-variable, one has

$$\lim_{n\to\infty} d(f^n(y, x_1), f^n(y, x_2)) = 0,$$

where d is a metric on X and $F^n(y, x) = (g^n(y), f^n(y, x))$. An illustrative example, of linear differential equations forced by the Lorenz equations, is given by Pecora and Carroll in [36]. We refer to [37] for an explanation of synchronization in a range of contexts.

The following result describes a similar effect for step skew product systems $F^+ \in \mathcal{S}$; the proof employs a measurable invariant graph for the natural extension F of F^+.

Theorem 2.4.1. *Let $F^+ \in \mathcal{S}$ and assume $L(0) > 0$ and $L(1) > 0$. Let $x_0, y_0 \in (0, 1)$. Then for ν^+-almost all ω,*

$$\lim_{n\to\infty} |f^n_\omega(x_0) - f^n_\omega(y_0)| = 0.$$

Proof. The proof of Theorem 2.3.1 gives the existence of an invariant measurable graph $\xi : \Sigma_2 \to I$ so that given any $x_0 \in (0, 1)$, one has that for ν-almost all ω,

$$\lim_{n\to\infty} |f^n_{\sigma^{-n}\omega}(x_0) - \xi(\omega)| = 0. \tag{2.4.1}$$

As ν is invariant for σ, this proves that $f^n_\omega(x_0)$ converges to $\xi(\sigma^n\omega)$ in probability. This gives the existence of a subsequence $n_k \to \infty$ as $k \to \infty$ so that

$$\lim_{k\to\infty} |f^{n_k}_\omega(x_0) - \xi(\sigma^{n_k}\omega)| = 0$$

(see e.g. [39, Theorem II.10.5]). We have thus obtained the weaker statement that for ν^+-almost all ω,

$$\liminf_{n\to\infty} |f^n_\omega(x_0) - \xi(\sigma^n\omega)| = 0.$$

We provide a sketch of the argument to show that this is also true with limit replacing the limit inferior, which would imply the theorem. The measure $(\mathrm{id}, \xi)\nu$ on $\Sigma_2 \times I$, with conditional measures $\delta_{\xi(\omega)}$ and marginal ν, is invariant for the natural extension $F : \Sigma_2 \times I \to \Sigma_2 \times I$ of F^+. It corresponds to an invariant measure $\nu^+ \times m$ for F^+ by Proposition 2.A.4 and the observation that $\xi(\omega)$ depends on $(\omega_n)_{-\infty}^{-1}$ only.

Lemma 2.4.2. *With respect to the measure $\nu^+ \times m$, the system F^+ has a negative Lyapunov exponent;*

$$\lambda = \sum_{i=1}^{2} p_i \int_I \ln(f_i'(x))\, dm(x) < 0.$$

Proof. One can follow the argument for [29, Theorem 7.1] (an analogue of [8, Theorem 4.2]). We describe the steps. A key idea is the use of the notion of relative entropy; the relative entropy $h(m_1|m_2)$ of a probability measure m_1 on I with respect to a probability measure m_2 on I is given by

$$h(m_1|m_2) = \sup_{\psi \in C^0(I)} \left(\ln \left(\int_I e^{\psi(x)}\, dm_1(x) \right) - \int_I \psi(x)\, dm_2(x) \right).$$

The following properties hold [15].

(i) $0 \leq h(m_1|m_2) \leq \infty$;

(ii) $h(m_1|m_2) = 0$ if and only if $m_1 = m_2$;

A relation between Lyapunov exponent and relative entropy can be derived for absolutely continuous stationary measures. The argument now involves maps with absolutely continuous noise with shrinking amplitude to approximate the fiber diffeomorphisms. Such a perturbed system admits an absolutely continuous stationary measure. One uses the relation between the Lyapunov exponent and relative entropy for this absolutely continuous stationary measure and considers the limit where the noise amplitude shrinks to zero.

Let ζ be a random variable with values in $[0,1]$ that is uniformly distributed. For small positive values of ε, let

$$f_{i,\zeta}(x) = (1-\varepsilon) f_i(x) + \zeta\varepsilon.$$

Note that

$$\zeta\varepsilon = f_{i,\zeta}(0) \leq f_{i,\zeta}(x) \leq f_{i,\zeta}(1) = 1 - \varepsilon + \zeta\varepsilon,$$

so that $f_{i,\zeta}$ maps $[0,1]$ into $[0,1]$, for each value of ζ in $[0,1]$. The iterated function system generated by the maps $f_{i,\zeta}$ has a stationary measure

$$m_\varepsilon = \sum_{i=1}^{2} p_i \int_0^1 f_{i,\zeta} m_\varepsilon \, d\zeta. \tag{2.4.2}$$

Note that m_ε is a fixed point of an operator $\mathcal{T}_\varepsilon$ where $\mathcal{T}_\varepsilon m_\varepsilon$ is defined by the right hand side of (2.4.2). One can show that m_ε has a smooth density [42]. Moreover, with $\mathcal{N}_c$ the closed set of probability measures considered in the proof of Lemma 2.3.4, one has that for for suitable values of α, c, q, $m_\varepsilon \in \mathcal{N}_c$ for all small positive ε. This is true since $\mathcal{T}_\varepsilon$ maps $\mathcal{N}_c$ into itself for suitable values of α, c, q. To see this follow the proof of Lemma 2.3.4 with $\mathcal{T}_\varepsilon$ replacing $\mathcal{T}$. The main calculation analogous to (2.3.3) is straightforward noting that $f_{i,\zeta}^{-1}([0,x)) \subset f_{i,0}^{-1}([0,x))$ for all

$\zeta \in [0,1]$:

$$\begin{aligned}
\mathcal{T}_\varepsilon m_\varepsilon\big([0,x)\big) &= \sum_{i=1}^{2} p_i \int_0^1 f_{i,\zeta} m_\varepsilon\big([0,x)\big)\, d\zeta = \sum_{i=1}^{2} p_i \int_0^1 m_\varepsilon\left(f_{i,\zeta}^{-1}[0,x)\right)\, d\zeta \\
&\leq \sum_{i=1}^{2} p_i \int_0^1 m_\varepsilon\left(f_{i,0}^{-1}[0,x)\right)\, d\zeta \leq \sum_{i=1}^{2} p_i m_\varepsilon\left([0, \frac{x}{\rho_i - \delta})\right) \\
&\leq \sum_{i=1}^{2} p_i c \left(\frac{x}{\rho_i - \delta}\right)^\alpha = c\left(\sum_{i=1}^{2} \frac{p_i}{(\rho_i - \delta)^\alpha}\right) x^\alpha \leq c x^\alpha.
\end{aligned}$$

A similar argument can be employed near the boundary point 1, for ε small.

The Lyapunov exponent for the stationary measure m_ε is given by

$$\lambda_\varepsilon = \sum_{i=1}^{2} p_i \int_0^1 \int_{\mathbb{I}} \ln(f_{i,\zeta}'(x))\, dm_\varepsilon(x) d\zeta. \tag{2.4.3}$$

Since m_ε has a smooth and bounded density, one can prove the relation (see [29, Proposition 7.2])

$$\lambda_\varepsilon = -\sum_{i=1}^{2} p_i \int_0^1 h(f_{i,\zeta} m_\varepsilon | m_\varepsilon)\, d\zeta. \tag{2.4.4}$$

By (i), $\lambda_\varepsilon \leq 0$. By (ii), $\lambda_\varepsilon = 0$ if and only if $f_{i,\zeta} m_\varepsilon = m_\varepsilon$ for all $\zeta \in [0,1]$ and $i = 1,2$. As the latter is not possible, $\lambda_\varepsilon < 0$.

Now take the limit $\varepsilon \to 0$. Then $m_\varepsilon \to m$ since $\mathcal{T}_\varepsilon$ is continuous and depends continuously on ε (compare Lemma 2.A.2), convergence is in $\mathcal{N}_c$, and m is the unique stationary measure in $\mathcal{N}_c$. From (2.4.3) one sees that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$ and we obtain $\lambda \leq 0$. Since the relative entropy is lower semi-continuous in ε as the supremum of continuous functionals, one finds from (2.4.4) that

$$0 \leq \sum_{i=1}^{2} p_i h(f_i m | m) \leq -\lambda.$$

This shows that $h(f_i m|m) = 0$ for $i = 1,2$ in case $\lambda = 0$. This is clearly not the case by ((ii)), as $f_1 m \neq f_2 m \neq m$. So we have $\lambda < 0$. $\square$

Because of this lemma, for ν-almost all ω, $\xi(\omega)$ from (2.4.1) has a stable manifold $W^s(\omega)$ that is an open neighborhood of $\xi(\omega)$ in I. To see this one can refer to general theory for nonuniformly hyperbolic systems as in [7], or apply reasoning as in Lemma 2.3.2. For each $x \in W^s(\omega)$,

$$\lim_{n\to\infty} |f_\omega^n(x) - \xi(\sigma^n \omega)| = 0.$$

Write

$$W^s(\omega) = (r^b(\omega), r^t(\omega)).$$

Then r^b and r^t are invariant. Hence $r^b > 0$, ν-almost everywhere, or $r^b = 0$, ν-almost everywhere, and likewise $r^t < 1$, ν-almost everywhere, or $r^t = 1$, ν-almost everywhere. We will derive a contradiction from the assumption that $r^t < 1$ or $r^b > 0$, ν-almost everywhere. Assume that e.g. $r^t < 1$, ν-almost everywhere. Write

$$r(\omega) = \inf\{x \in I \mid \lim_{n\to\infty} f_\omega^{-n}(x) = 1\}. \tag{2.4.5}$$

As $L(1) > 0$, we have $r(\omega) < 1$ for ν-almost all $\omega \in \Sigma_2$, compare Lemma 2.3.2. Since the graphs of r^t and r are invariant graphs and also $r^t < 1$, we have $r \geq r^t > \xi$, ν-almost everywhere.

The measure $\mu = (\mathrm{id}, r)\nu$ on $\Sigma_2 \times I$ with conditional measures $\delta_{r(\omega)}$ and marginal ν defines an invariant measure for F. It follows from the expression (2.4.5) that $r(\omega)$ depends on the past $\omega^- = (\omega_n)_{-\infty}^{-1}$ only. Consequently, μ is a product measure of the form $\nu^+ \times \vartheta$ on $\Sigma_2^+ \times (\Sigma_2^- \times I)$. With Π the natural projection $\Sigma_2 \times I \to \Sigma_2^+ \times I$, we find that the F^+-invariant measure $\Pi\mu$ is a product measure $\Pi\mu = \nu^+ \times \hat{m}$ on $\Sigma_2^+ \times I$. By Lemma 2.A.3, $\hat{m}$ is a stationary measure. Since $r > \xi$, ν-almost everywhere, Proposition 2.A.4 gives that $m \neq \hat{m}$. Lemma 2.3.5 however prohibits the existence of two different stationary measures with support in $(0, 1)$. The contradiction is derived, establishing that $W^s(\omega) = (0, 1)$ for ν-almost all $\omega \in \Sigma_2$. $\square$

Under the conditions of Theorem 2.4.1, the proof of Theorem 2.3.1 shows that

$$\lim_{n\to\infty} |f^n_{\sigma^{-n}\omega}(x) - \xi(\omega)| = 0$$

for ν-almost all $\omega \in \Sigma_2$ and any $x \in (0, 1)$. This convergence is called pullback convergence. The proof of Theorem 2.4.1 shows that

$$\lim_{n\to\infty} |f^n_\omega(x) - \xi(\sigma^n\omega)| = 0$$

for ν-almost all $\omega \in \Sigma_2$ and any $x \in (0, 1)$. This convergence is called forward convergence. So in this case both pullback and forward convergence to ξ holds. In general however forward convergence is not a consequence of pullback convergence. The next section provides an example, involving a zero Lyapunov exponent, with pullback convergence but not forward convergence. See in particular Section 2.5.1. Section 2.6 contains a related example, related by going to the inverse skew product system, with forward convergence but not pullback convergence. We refer to [30] for more discussion on conditions for convergence in nonautonomous and skew product systems.

We finish with some pointers to further literature. In [3, 27, 14, 43] synchronization results, similar to Theorem 2.4.1, for skew product systems with circle diffeomorphisms as fiber maps are treated without employing negativity of Lyapunov exponents. Motivated by Lemma 2.4.2 for example, one may wonder about other invariant measures than those with Bernoulli measure as marginal. Reference [20] considers, in this direction, the existence of nonhyperbolic measures for step skew product systems with circle fibers.

2.5 On-off intermittency

Intermittency in a dynamical system stands for dynamics that exhibits alternating phases of different characteristics. Typically, intermittent dynamics alternates time series close to equilibrium with bursts of global dynamics [9]. In our context, we say that a step skew product system $F^+ \in \mathcal{S}$ displays intermittency if the following holds for any sufficiently small neighborhood U of 0:

1. For all $x \in (0,1)$ and ν^+-almost all $\omega \in \Sigma_2^+$:

$$\lim_{n\to\infty} \frac{1}{n} \left|\{0 \le i < n \ ; \ f_\omega^i(x) \in U\}\right| = 1;$$

2. For all $x \in (0,1)$ and ν^+-almost all $\omega \in \Sigma_2^+$, $f_\omega^n(x) \notin U$ for infinitely many n.

Here, for a finite set S, we write $|S|$ for its cardinality.

This kind of intermittency that involves a weakly unstable invariant set, here $\Sigma_2^+ \times \{0\} \subset \Sigma_2^+ \times I$, has been called on-off intermittency [38, 21]. The occurrence of intermittency in iterated function systems of logistic maps with zero Lyapunov exponent at the fixed point in 0 is treated in [5, 6]. See also [11] for a study of specific interval diffeomorphisms over expanding circle maps.

In this section we will discuss on-off intermittency for step skew product systems $F^+ \in \mathcal{S}$. Throughout we assume that both diffeomorphisms f_1, f_2 are picked with probability $1/2$. This is for convenience, we expect the more general case with probabilities p_1, p_2 to go along the same lines. The following two theorems, Theorems 2.5.1 and 2.5.3, demonstrate that $F^+ \in \mathcal{S}$ with $L(0) = 0$ and $L(1) > 0$ displays intermittency. Figure 2.4 illustrates a typical time series.

Lamperti, in a sequence of papers [31, 32, 33], developed a general theory of recurrence for nonhomogeneous random walks on the half-line. His results may be used to prove on-off intermittency in our context, see in particular [33, Theorems 3.1 and 4.1]. We will get it by calculating bounds on stopping times, using C^2 differentiability of the generating diffeomorphisms.

Theorem 2.5.1. *Let $F^+ \in \mathcal{S}$ and assume $L(0) = 0$. Let $0 < \beta$ be small and $x_0 \in (0,1)$. Then for ν^+-almost every $\omega \in \Sigma_2^+$, $f_\omega^n(x_0)$ is in $[\beta, 1]$ for infinitely many values of n.*

Proof. We follow the proof of [6, Theorem 1]. Given $x_0 \in \mathbb{I}$ and $\omega \in \Sigma_2^+$, write

$$x_n = x_n(\omega) = f_\omega^n(x_0).$$

It suffices to show that for $x_0 < \beta$,

$$\nu^+(\{\omega \in \Sigma_2^+ \ \mid \ x_n \ge \beta \text{ for some } n \ge 1\}) = 1.$$

Let

$$u_n = -\ln(x_n).$$

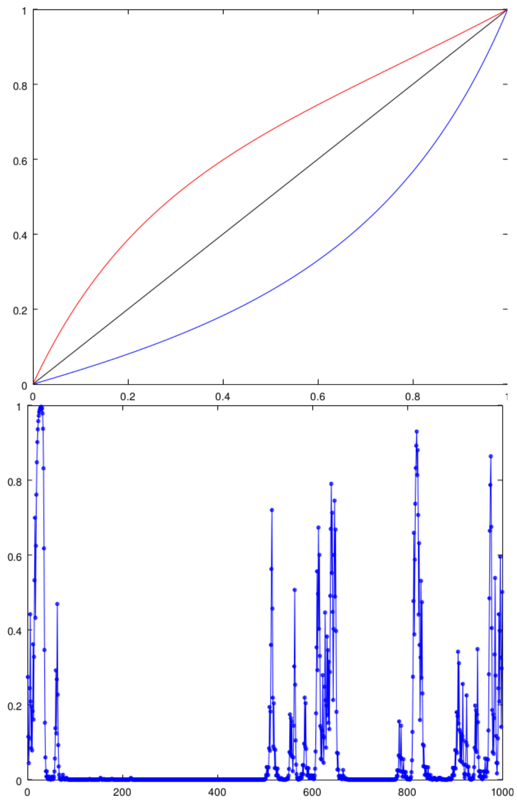

Figure 2.4: *The left frame depicts the graphs of* $x \mapsto f_i(x) = g_i(x)(1-p(x))$, $i = 1, 2$, *with* $g_i(x)$ *as in* (2.2.2), (2.2.3) *and* $p(x) = \frac{3}{10}x(1-x)$. *The corresponding step skew product system has a zero Lyapunov exponent along* $\Sigma_2^+ \times \{0\}$ *and a positive Lyapunov exponent along* $\Sigma_2^+ \times \{1\}$. *The right frame shows a time series for the iterated function system generated by these diffeomorphisms.*

We wish to show that for $u_0 > K = -\ln(\beta)$,

$$\nu^+(\{\omega \in \Sigma_2^+ \mid u_n \leq K \text{ for some } n \geq 1\}) = 1.$$

Write $f_i(x) = c_i x/(1 + t_i(x))$ with $t_i(x) = O(x)$ as $x \to 0$. Taking logarithms of $x_{n+1} = f_{\omega_n}(x_n)$ we get

$$u_{n+1} = d_{\omega_n} + u_n + \ln(1 + t_{\omega_n}(e^{-u_n})),$$

with $d_i = -\ln(c_i)$. Consider the stopping time

$$T = \inf\{n \geq 1 \mid u_n \leq K\}$$

(with $T = \infty$ if $u_n > K$ for all n) and write

$$z_n = \ln(u_{n \wedge T}),$$

where $n \wedge T = \min\{n, T\}$. We claim that z_n is a supermartingale;

Lemma 2.5.2.

$$\int_C z_{n+1}(\omega)\, d\nu^+(\omega) \leq \int_C z_n(\omega)\, d\nu^+(\omega)$$

for cylinders $C = C^{0,\ldots,n-1}_{\omega_0,\ldots,\omega_{n-1}}$.

Proof. On C, $z_n(\omega)$ is constant. As further $c_{\omega_{n+1}}$ is independent of c_{ω_n}, it suffices to consider $n = 0$ and to prove

$$\int_{\Sigma_2^+} z_1(\omega)\, d\nu^+(\omega) \leq \ln(u_0)$$

for u_0 large enough. Denote

$$h(u_0) = \int_{\Sigma_2^+} z_1\, d\nu^+(\omega) - \ln(u_0).$$

The zero Lyapunov exponent, $L(0) = 0$, implies $\int_{\Sigma_2^+} d_{\omega_0}\, d\nu^+(\omega) = 0$. Using this,

$$\begin{aligned}
h(u) &= \int_{\Sigma_2^+} \ln\left(d_{\omega_0} + u + \ln(1 + t_{\omega_0}(e^{-u}))\right) - \ln(u)\, d\nu^+(\omega) \\
&= \int_{\Sigma_2^+} \ln\left(\frac{d_{\omega_0} + u + \ln(1 + t_{\omega_0}(e^{-u}))}{u}\right) d\nu^+(\omega) \\
&= \int_{\Sigma_2^+} \ln\left(\frac{d_{\omega_0} + u + \ln(1 + t_{\omega_0}(e^{-u}))}{u}\right) - \frac{d_{\omega_0}}{u}\, d\nu^+(\omega) \\
&= \int_{\Sigma_2^+} \ln\left(\frac{d_{\omega_0}}{u} + 1 + \frac{\ln(1 + t_{\omega_0}(e^{-u}))}{u}\right) - \frac{d_{\omega_0}}{u}\, d\nu^+(\omega).
\end{aligned}$$

By developing the integrand of the last expression in a Taylor expansion, this gives

$$h(u) = -\frac{1}{2}\int_{\Sigma_2^+} \left(\frac{d_{\omega_0}}{u}\right)^2 + o\left(\frac{1}{u^2}\right) d\nu^+(\omega),\ u \to \infty.$$

So

$$\limsup_{u\to\infty} h(u)u^2 < 0,$$

implying that there exists $\bar{u}$ so that $h(u) < 0$ for $u \geq \bar{u}$. □

Now that Lemma 2.5.2 gives that z_n is a nonnegative supermartingale, by Doob's supermartingale convergence theorem, see e.g. [39, Section VII.4],

$$\lim_{n\to\infty} z_n(\omega) < \infty \tag{2.5.1}$$

for ν^+-almost all $\omega \in \Sigma_2^+$. Let

1. $B_1 = \{\omega \in \Sigma_2^+ \mid z_\infty = \lim_{n\to\infty} z_n < \infty\}$;

2. $B_2 = \{\omega \in \Sigma_2^+ \mid T = \infty\}$.

We must prove that $\nu^+(B_2) = 0$. On $B_1 \cap B_2$, $z_n \to z_\infty$ and thus $x_n \to x_\infty \in (0,1)$ as $n \to \infty$. This is impossible as both $f_1(x_\infty) \neq x_\infty$ and $f_2(x_\infty) \neq x_\infty$. So $B_1 \cap B_2 = \emptyset$. By (2.5.1), $\nu^+(B_1) = 1$. Hence $\nu^+(B_2) = 0$. □

If one assumes $L(1) > 0$, then a similar, in fact simpler, argument shows that for β small and $x_0 \in (0,1)$, for ν^+-almost all ω one finds $x_n = f_\omega^n(x_0)$ in $[0, 1-\beta]$ for infinitely many values of n.

Theorem 2.5.3. *Consider $F^+ \in \mathcal{S}$ and assume $L(0) = 0$ and $L(1) > 0$. Let $0 < \beta < 1$ and $x_0 \in (0,1)$. Then for ν^+-almost every $\omega \in \Sigma_2^+$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_{[0,\beta)}(f_\omega^i(x_0)) = 1. \tag{2.5.2}$$

Proof. The reasoning is inspired by [6, Theorem 4]. Consider

$$x_n = x_n(\omega) = f_\omega^n(x_0)$$

and

$$y_n = \ln(x_n/(1-x_n)).$$

We denote

$$y_{n+1} = h_{\omega_n}(y_n).$$

For β small, $K = \ln(\beta/(1-\beta))$ is a large negative number. For definiteness assume $x_0 \le \beta$, i.e. $y_0 \le K$. Define stopping times $T_0 = 0$,

$$\begin{aligned} T_{2k+1} &= \inf\{n \in \mathbb{N} \mid n > T_{2k} \text{ and } y_n > K\}, \\ T_{2k} &= \inf\{n \in \mathbb{N} \mid n > T_{2k-1} \text{ and } y_n \le K\}, \end{aligned}$$

see Figure 2.5. Let

$$\begin{aligned} \eta_k &= |[T_{2k-2}, T_{2k-1})| = T_{2k-1} - T_{2k-2}, \\ \xi_k &= |[T_{2k-1}, T_{2k})| = T_{2k} - T_{2k-1} \end{aligned}$$

be the duration of subsequent iterates with $y_n \le K$ and the duration of subsequent iterates with $y_n > K$, respectively.

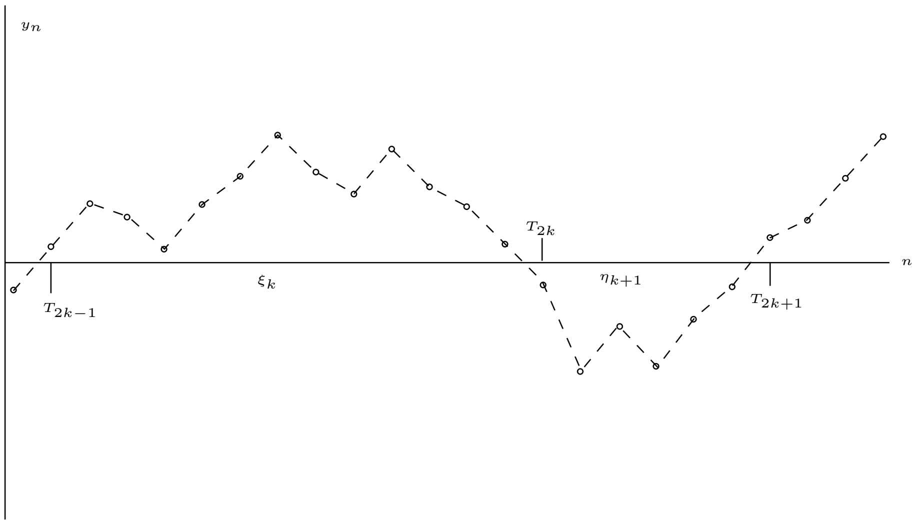

Figure 2.5: *A sequence of stopping times is defined to label subsequent iterates where y_n leaves $(-\infty, K]$ or (K, ∞).*

Lemmas 2.5.4 and 2.5.6 determine bounds for the expectation of the stopping times η_k (which is shown to be infinite) and ξ_k (which is shown to be finite).

Lemma 2.5.4.

$$\int_{\Sigma_2^+} \eta_k(\omega)\, d\nu^+(\omega) = \infty; \tag{2.5.3}$$

η_k has infinite expectation.

Proof. A smooth conjugation brings f_1 near 0 to the linear map $x \mapsto x/d$ for some $d > 1$. More formally, there is a local diffeomorphism $h : U \to \mathbb{R}$ with U a neighborhood of 0 and $h(0) = 0$, so that $h^{-1} \circ f_1 \circ h(x) = x/d$. Replace f_1 by

$h^{-1} \circ f_1 \circ h$ and likewise f_2 by $h^{-1} \circ f_2 \circ h$, near 0. That is, we may assume that on a neighborhood of 0 that contains $[0, \beta]$,

$$\begin{aligned} f_1(x) &= x/d, \\ f_2(x) &= dx(1 + r(x)), \end{aligned}$$

for some smooth function $r(x) = O(|x|)$, $x \to 0$ (observe that $f_1'(0) = 1/f_2'(0)$ by $L(0) = 0$).

Let $\tilde{z}_n = \ln(x_n)$ map $(0, \beta]$ to $(-\infty, \tilde{L}]$ with $\tilde{L} = \ln(\beta)$. Then $x_{n+1} = f_{\omega_n}(x_n)$ becomes

$$\tilde{z}_{n+1} = \begin{cases} \tilde{z}_n - \ln(d), & \text{if } \omega_n = 1, \\ \tilde{z}_n + \ln(d) + \ln(1 + r(e^{\tilde{z}_n})), & \text{if } \omega_n = 2. \end{cases}$$

An additional rescaling $z_n = \tilde{z}_n / \ln(d)$ conjugates this iterated function to

$$z_{n+1} = \begin{cases} z_n - 1, & \text{if } \omega_n = 1, \\ z_n + 1 + \ln(1 + r(e^{z_n \ln(d)}))/\ln(d), & \text{if } \omega_n = 2. \end{cases} \tag{2.5.4}$$

Write $L = \tilde{L}/\ln(d)$; we consider z_n on $(-\infty, L]$.

Let $g > 0$; g will be chosen large in the sequel. The term $\ln(1+r(e^{z_n \ln(d)}))/\ln(d)$ may be bounded from above by $Ce^{-g \ln(d)}$ on intervals $(-\infty, L-g]$, for some $C > 0$. On $(-\infty, L-g] \subset (-\infty, L]$ we compare the random walk z_n with the random walk

$$v_{n+1} = \begin{cases} v_n - 1, & \text{if } \omega_n = 1, \\ v_n + 1 + Ce^{-g \ln(d)}, & \text{if } \omega_n = 2. \end{cases}$$

Given $z_0 = v_0 \in [L - g - 1, L - g)$, we define stopping times

$$\begin{aligned} T_z &= \min\{n \in \mathbb{N} \mid z_n \geq L - g\}, \\ T_v &= \min\{n \in \mathbb{N} \mid v_n \geq L - g\}. \end{aligned}$$

If $z_0, \ldots, z_n \in (-\infty, L - g)$, then $z_i \leq v_i$ for all $0 \leq i \leq n+1$. Therefore, for each $\omega \in \Sigma_2^+$,

$$T_z(\omega) \geq T_v(\omega).$$

By Wald's identity, see e.g. [39, Section VII.2],

$$\begin{aligned} \int_{\Sigma_2^+} T_v(\omega)\, d\nu^+(\omega) &= \frac{2}{Ce^{-g \ln(d)}} \int_{\Sigma_2^+} v_{T_v(\omega)} - v_0 \, d\nu^+(\omega) \\ &\geq ce^{g \ln(d)} \end{aligned}$$

for some $c > 0$, and hence

$$\int_{\Sigma_2^+} T_z(\omega)\, d\nu^+(\omega) \geq ce^{g \ln(d)}. \tag{2.5.5}$$

Let $\alpha > 0$ be so that

$$z_{n+1} \leq z_n + 1 + \alpha \tag{2.5.6}$$

for $L - g \leq z_n \leq L$. Note that we may take α to be small if L is large. Consider the random walk given by (2.5.4) with initial point $z_0 \in (L-1, L]$. Define the stopping time

$$T_g = \min\{n > 0 \mid z_n < L - g \text{ or } z_n > L\}.$$

Lemma 2.5.5. *For $\alpha > 0$ small enough, there is $c, r^* = r^*(\alpha) < 0$, so that*

$$\nu^+(\{\omega \in \Sigma_2^+ \mid z_{T_g} < L - g\}) \geq ce^{gr^*}.$$

Here $r^*(\alpha) \to 0$ as $\alpha \to 0$.

We finish the proof of Lemma 2.5.4 using this lemma, and we then prove Lemma 2.5.5. Consider the following reasoning. Start with a point $z_{T_{2k}} \in (L - 1, L]$. Then some iterate of $z_{T_{2k}}$ will have left $[L-g, L]$, either through the right boundary point L or, with probability determined by Lemma 2.5.5, through the left boundary point $L - g$. In the latter case there will be a return time to $[L-g, L]$ after which a further iterate may leave through the right boundary point L. Consequently, combining (2.5.5) and Lemma 2.5.5,

$$\int_{\Sigma_2^+} \eta_k(\omega)\, d\nu^+(\omega) \geq ce^{gr^*} e^{g \ln(d)} \tag{2.5.7}$$

for some $c > 0$. For L sufficiently large, α is small enough to ensure $e^{r^*} e^{\ln(d)} > 1$, because $r^*(\alpha) \to 0$ as $\alpha \to 0$. Then the right hand side of (2.5.7) goes to infinity as $g \to \infty$. This concludes the proof of Lemma 2.5.4. $\square$

Proof of Lemma 2.5.5. Consider the random walk

$$u_{n+1} = \begin{cases} u_n - 1, & \text{if } \omega_n = 1, \\ u_n + 1 + \alpha, & \text{if } \omega_n = 2, \end{cases}$$

with $u_0 \in (-1, 0]$. Define the stopping time

$$U_g = \min\{n > 0 \mid u_n < -g \text{ or } u_n > 0\}.$$

By (2.5.6) we have

$$\nu^+(\{\omega \in \Sigma_2^+ \mid z_{T_g} < L - g\}) \geq \nu^+(\{\omega \in \Sigma_2^+ \mid u_{U_g} < -g\})$$

and hence it suffices to prove the estimate

$$\nu^+(\{\omega \in \Sigma_2^+ \mid u_{U_g} < -g\}) \geq ce^{gr^*}.$$

Write ζ_n for the steps $u_n - u_{n-1}$; $\zeta_n = -1$ or $\zeta_n = 1+\alpha$ both with probability 1/2. Write $S_n = \zeta_1 + \cdots + \zeta_n = u_n - u_0$ and consider the function

$$G_n = e^{r^* S_n},$$

where $r^* < 0$ is the solution of

$$\frac{1}{2}e^{-r^*} + \frac{1}{2}e^{r^*(1+\alpha)} = 1.$$

One can check that this equation has a unique solution $r^* < 0$ with $r^* \to 0$ as $\alpha \to 0$. Now G_n is a martingale as

$$\begin{aligned}
\int_{C^{0,\dots,n-1}_{\omega_0,\dots,\omega_{n-1}}} e^{r^* S_n}\, d\nu^+(\omega) &= \int_{C^{0,\dots,n-1}_{\omega_0,\dots,\omega_{n-1}}} e^{r^* S_{n-1}} e^{r^*\zeta_n}\, d\nu^+(\omega) \\
&= e^{r^* S_{n-1}} \int_{C^{0,\dots,n-1}_{\omega_0,\dots,\omega_{n-1}}} e^{r^*\zeta_n}\, d\nu^+(\omega) \\
&= e^{r^* S_{n-1}} \left(\int_{C^{0,\dots,n}_{\omega_0,\dots,\omega_{n-1},1}} e^{-r^*}\, d\nu^+(\omega) + \int_{C^{0,\dots,n}_{\omega_0,\dots,\omega_{n-1},2}} e^{r^*(1+\alpha)}\, d\nu^+(\omega) \right) \\
&= e^{r^* S_{n-1}} \int_{C^{0,\dots,n-1}_{\omega_0,\dots,\omega_{n-1}}} \frac{1}{2}e^{-r^*} + \frac{1}{2}e^{r^*(1+\alpha)}\, d\nu^+(\omega) \\
&= \int_{C^{0,\dots,n-1}_{\omega_0,\dots,\omega_{n-1}}} e^{r^* S_{n-1}}\, d\nu^+(\omega).
\end{aligned}$$

By Doob's optional stopping theorem, see e.g. [39, Theorem VII.2.2],

$$\int_{\Sigma_2^+} e^{r^* S_{U_g}}\, d\nu^+(\omega) = \int_{\Sigma_2^+} e^{r^* S_0}\, d\nu^+(\omega) = 1.$$

This gives

$$\int_{\Sigma_2^+} e^{r^* u_{U_g}}\, d\nu^+(\omega) = e^{r^* u_0}.$$

Observe $u_{U_g} \in [-g-1, -g)$ or $u_{U_g} \in (0, 1+\alpha]$. Let

$$A = \nu^+(\{\omega \in \Sigma_2^+ \mid u_{U_g} < -g\})$$

be the probability that $u_{U_g} < -g$. Write

$$\int_{\Sigma_2^+} e^{r^* u_{U_g}}\, d\nu^+(\omega) = A e^{-gr^*} e^{-c_1 r^*} + (1-A) e^{c_2 r^*},$$

where

$$\begin{aligned}
e^{-gr^*} e^{-c_1 r^*} &= \frac{1}{A} \int_{\{\omega\in\Sigma_2^+ \mid u_{U_g} < -g\}} e^{r^* u_{U_g}}\, d\nu^+(\omega), \\
e^{c_2 r^*} &= \frac{1}{1-A} \int_{\{\omega\in\Sigma_2^+ \mid u_{U_g} > 0\}} e^{r^* u_{U_g}}\, d\nu^+(\omega).
\end{aligned}$$

In these expressions, $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1+\alpha$. A moment of thought gives that $c_2 > 0$ ($c_2 = 0$ can only occur if points leave $[-g, 0]$ through the right boundary point 0, but the initial point $u_0 \in (-1, 0]$ is mapped with probability 1/2 to a point in $(\alpha, 1+\alpha]$). We obtain

$$A\left(e^{-gr^*}e^{-c_1 r^*} - e^{c_2 r^*}\right) = e^{u_0 r^*} - e^{c_2 r^*},$$

where this last number is positive. □

Similar arguments prove the following lemma.

Lemma 2.5.6.

$$\int_{\Sigma_2^+} \xi_k(\omega)\, d\nu^+(\omega) < \infty; \tag{2.5.8}$$

ξ_k has finite expectation.

Proof. Recall that $x_{n+1} = f_{\omega_n}(x_n)$ on $\mathbb{I}$ is conjugate to $y_{n+1} = h_{\omega_n}(y_n)$ on $\mathbb{R}$ through $y_n = \ln(x_n/(1-x_n))$. We split iterates of y_n in $[K, \infty)$ into two sets, namely iterates in $[K, \tilde{K}]$ and iterates in $(\tilde{K}, \infty)$, for some positive and large $\tilde{K}$. Near $x = 1$, write $f_i(x) = 1 - a_i(1-x)(1+r_i(1-x))$ with $a_i > 0$ and $r_i(u) = O(u)$, $u \to 0$. The positive Lyapunov condition $L(1) > 0$ means that $\ln(a_1) + \ln(a_2) > 0$. Calculate

$$\begin{aligned} y_{n+1} = y_n - \ln(a_{\omega_n}) + {} & \\ & \ln(1-(1-x_n)) + \ln(1+r_{\omega_n}(1-x_n)) + \ln(1-a_{\omega_n}(1-x_n)(1+r_{\omega_n}(1-x_n))), \end{aligned}$$

where $1 - x_n = 1/(1+e^{y_n})$. From this expression it is easily seen for any $\varepsilon > 0$ one can pick $\tilde{K}$ large, so that for $y_n > \tilde{K}$,

$$y_{n+1} \leq y_n - \ln(a_{\omega_n}) + \varepsilon.$$

Pick ε small enough so that $-\ln(a_1) - \ln(a_2) + 2\varepsilon < 0$.

For $z_0 \in (\tilde{K}, h_2(\tilde{K})]$ and $z_{n+1} = h_{\omega_n}(z_n)$, let

$$T_{\tilde{K}} = \min\{n \in \mathbb{N} \mid z_n \leq \tilde{K}\}$$

be the stopping time to leave $(\tilde{K}, \infty)$. As in the proof of Lemma 2.5.4 one shows that the expectation of $T_{\tilde{K}}$ is finite. To provide the argument, consider the random walk

$$u_{n+1} = u_n - \ln(a_{\omega_n}) + \varepsilon$$

starting at $u_0 = z_0$ and let

$$T_u = \min\{n \in \mathbb{N} \mid u_n \leq \tilde{K}\}.$$

Then $T_{\tilde{K}} \le T_u$. By Wald's identity, $\int_{\Sigma_2^+} T_u(\omega)\, d\nu^+(\omega) < \infty$ and hence

$$\int_{\Sigma_2^+} T_{\tilde{K}}(\omega)\, d\nu^+(\omega) < \infty. \tag{2.5.9}$$

After these preparations we define the first return map $g_\omega : (-\infty, \tilde{K}] \to (-\infty, \tilde{K}]$,

$$g_\omega(y) = h_\omega^{R(\omega,y)}(y),$$

where

$$R(\omega, y) = \min\{n \ge 1 \mid h_\omega^n(y) \le \tilde{K}\}.$$

By (2.5.9), R has finite expectation. In fact, there is $C > 0$ so that for each $y \in [K, \tilde{K}]$,

$$\int_{\Sigma_2^+} R(\omega, y)\, d\nu^+(\omega) \le C. \tag{2.5.10}$$

The next step is to show that

$$T_K = \min\{n \in \mathbb{N} \mid g_\omega^n(y) < K\}$$

for $y \in [K, \tilde{K}]$ has finite expectation. Consider the skew product $G : \Sigma_2^+ \times (-\infty, \tilde{K}] \to \Sigma_2^+ \times (-\infty, \tilde{K}]$,

$$G(\omega, y) = (\sigma^{R(\omega,y)}\omega, h_\omega^{R(\omega,y)}(y)) = (\sigma^{R(\omega,y)}\omega, g_\omega(y)).$$

Let π be the projection $\pi(\omega, y) = \omega_0$ from $\Sigma_2^+ \times \mathbb{R}$ onto $\{1, 2\}$. Given $(\omega, y) \in \Sigma_2^+ \times [K, \tilde{K}]$ we obtain a sequence $\rho \in \Sigma_2^+$ given by

$$\rho_i = \pi G^i(\omega, y).$$

It follows from the construction that as the sequence $(\omega_i)_0^\infty$ is independent and identically distributed, also $(\rho_i)_0^\infty$ is independent and identically distributed with the same distribution: probability $1/2$ for both symbols $1, 2$. Because $f_1(x) < x$, we find $h_1(y) < y$ and thus that there is a number $l < 0$ with

$$h_1(y) < y + l,$$

for $y \in [K, \tilde{K}]$. Hence, for any $y \in [K, \tilde{K}]$ and $N = \lceil (K - \tilde{K})/l \rceil$ we will have $g_1^N(y) = h_1^N(y) < K$. The stopping time T_K is therefore smaller than the stopping time

$$\min\{n \in \mathbb{N} \mid \rho_i = 1 \text{ for } n - N < i \le n\}.$$

Note that the expected number of throws of symbols $1, 2$ that lead to N consecutive 1's is finite. (In fact it equals $2^{N+1} - 2$. It is easily bounded by N times the expectation of the first number j so that $\omega_i = 1$ for $jN \le i < j(N + 1)$; the latter is a geometric distribution with expectation 2^N). So the expectation of the stopping time T_K is finite;

$$\int_{\Sigma_2^+} T_K(\omega)\, d\nu^+(\omega) < \infty. \tag{2.5.11}$$

Finally we combine (2.5.10) and (2.5.11): the formula

$$\xi_k(\omega) = \sum_{n=0}^{T_K(\omega)-1} R(G^n(\omega, y_{T_{2k-1}}))$$

implies that

$$\begin{aligned}\int_{\Sigma_2^+} \xi_k(\omega)\, d\nu^+(\omega) &\leq C \int_{\Sigma_2^+} T_K(\omega)\, d\nu^+(\omega)\\ &< \infty.\end{aligned}$$

This proves Lemma 2.5.6. □

We can now finish the proof of Theorem 2.5.3. Define for $n \in [T_{2k}, T_{2k+1})$,

$$N_\eta(n) = k,\ N_\xi(n) = k$$

and

$$\tilde{\eta}(n) = n + 1 - T_{2k},\ \tilde{\xi}(n) = 0,$$

so that $\tilde{\eta}$ counts the number of iterates from T_{2k} on where $y_n \leq K$. Likewise define for $n \in [T_{2k+1}, T_{2k+2})$,

$$N_\eta(n) = k+1,\ N_\xi(n) = k$$

and

$$\tilde{\eta}(n) = 0,\ \tilde{\xi}(n) = n + 1 - T_{2k+1}.$$

So $\tilde{\xi}$ counts the number of iterates from T_{2k+1} on where $y_n > K$.

Finally calculate

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_{[0,\beta)}(f_\omega^i(x_0)) &= \frac{1}{n}\left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1)\right)\\
&= \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1)\right) \Bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1) + \sum_{k=1}^{N_\xi(n-1)} \xi_k + \tilde{\xi}(n-1)\right)\\
&= \left(1 + \left(\sum_{k=1}^{N_\xi(n-1)} \xi_k + \tilde{\xi}(n-1)\right) \Bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k + \tilde{\eta}(n-1)\right)\right)^{-1}\\
&\geq \left(1 + \left(\sum_{k=1}^{N_\xi(n-1)+1} \xi_k\right) \Bigg/ \left(\sum_{k=1}^{N_\eta(n-1)} \eta_k\right)\right)^{-1}.
\end{aligned}$$

By (2.5.3) and (2.5.8), the last term goes to 1 for ν^+-almost all ω, as $n \to \infty$ (note that $N_\eta(n-1) - N_\xi(n-1) \leq 1$). □

The next theorem is an immediate consequence of Theorem 2.5.3.

Theorem 2.5.7. *Let $F^+ \in \mathcal{S}$ and assume $L(0) \leq 0$ and $L(1) > 0$. Then the only ergodic stationary measures are the delta measures at 0 and 1.*

Proof. We will only treat the case $L(0) = 0$ and $L(1) > 0$. Suppose there is an ergodic stationary measure m with support in $(0,1)$. By Lemma 2.A.3, $\nu^+ \times m$ is an ergodic invariant measure for F^+. By Birkhoff's ergodic theorem, for $\nu^+ \times m$-almost every (ω, x), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i_\omega(x)} = m. \tag{2.5.12}$$

By Fubini's theorem, there is a subset of I of full m-measure, so that in any $\Sigma_2^+ \times \{x\}$ with x from this subset, there is a set of full ν^+-measure for which (2.5.12) holds. This however contradicts (2.5.2), since that holds for all $\beta > 0$ and applies to all $x \in I$. □

The type of reasoning to prove Theorem 2.5.3 can be used to obtain the following result on iterated functions systems with zero Lyapunov exponents at both end points.

Theorem 2.5.8. *Consider $F^+ \in \mathcal{S}$ and assume $L(0) = L(1) = 0$. Let $0 < \beta < 1$ and $x_0 \in (0,1)$. Then for ν^+-almost every $\omega \in \Sigma_2^+$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{[\beta,1-\beta]}(f^i_\omega(x_0)) = 0.$$

Figure 2.1 illustrates a time series of the symmetric random walk, to which this theorem applies.

2.5.1 Pullback convergence

Theorem 2.5.1 implies that forward convergence of $f^n_\omega(x)$ to 0 does not hold: it is not true that for ν^+-almost all $\omega \in \Sigma_2^+$, $f^n_\omega(x) \to 0$ as $n \to \infty$. The next result stipulates that pullback convergence to 0 does hold. See also [4, Section 9.3.4] for a related example where pullback convergence does not imply forward convergence, in a context of stochastic differential equations.

Theorem 2.5.9. *Let $F^+ \in \mathcal{S}$ and suppose $L(0) = 0$ and $L(1) > 0$. Take $x \in (0,1)$. Then for ν-almost all $\omega \in \Sigma_2$,*

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x) = 0. \tag{2.5.13}$$

Proof. We reformulate the theorem to the following equivalent statement: for ν-almost all $\omega \in \Sigma_2$, and for all $y \in (0,1)$,

$$\lim_{n\to\infty} f_\omega^{-n}(y) = 1. \tag{2.5.14}$$

Equivalence of the statements (2.5.13) and (2.5.14) follows from the monotonicity of the interval diffeomorphisms: $f_\omega^{-n}(y) > x$ precisely if $f_{\sigma^{-n}\omega}^n(x) < y$ and thus for $\varepsilon_1, \varepsilon_2$ small positive numbers, $f_\omega^{-n}(\varepsilon_1) > 1-\varepsilon_2$ precisely if $f_{\sigma^{-n}\omega}^n(1-\varepsilon_2) < \varepsilon_1$.

To prove (2.5.14), consider

$$u(\omega) = \inf\{y \in I \mid \lim_{i\to\infty} f_\omega^{-i}(y) = 1\}. \tag{2.5.15}$$

As $L(1) > 0$, by Lemma 2.3.2 we know that u exists and $u < 1$, ν-almost everywhere. Since u is invariant we get that either $u > 0$, ν-almost everywhere, or $u = 0$, ν-almost everywhere. Assume that u is not identically 0. The measure $\mu = (\mathrm{id}, u)\nu$ on $\Sigma_2 \times I$ with conditional measures $\delta_{u(\omega)}$ and marginal ν on Σ_2, defines an invariant measure for F.

Denote by Π the natural projection $\Sigma_2 \times I \to \Sigma_2^+ \times I$, where $\Sigma_2 = \Sigma_2^- \times \Sigma_2^+$. Expression (2.5.15) gives that $u(\omega)$ depends on the past $\omega^- = (\omega_i)_{-\infty}^{-1}$ only. Therefore, the measure μ is a product measure $\nu^+ \times \vartheta$ on $\Sigma_2^+ \times \left(\Sigma_2^- \times I\right)$. The projection $\Pi\mu$ is therefore a product measure $\nu^+ \times m$ on $\Sigma_2^+ \times I$. That is, μ corresponds to an invariant measure $\nu^+ \times m$ for F^+, see Proposition 2.A.4. Here m is a stationary measure by Lemma 2.A.3. Since $0 < u < 1$, ν-almost everywhere, m assigns positive measure to $(0,1)$.

By Theorem 2.5.7, the only stationary measures are convex combinations of delta measures at 0 and 1. We have obtained a contradiction and proven (2.5.14) and hence the theorem. □

2.5.2 Central limit theorem

Under the assumptions of Theorem 2.5.9, its conclusion that $f_{\sigma^{-n}\omega}^n(x) \to 0$ for ν-almost all ω, implies that $f_{\sigma^{-n}\omega}^n(x)$ converges to 0 in probability. By σ-invariance of ν, $f_\omega^n(x)$ converges to 0 in probability. Hence, for any $a \in (0,1)$,

$$\lim_{n\to\infty} \nu^+(\{\omega \in \Sigma_2^+ \mid f_\omega^n(x) \le a\}) = 1.$$

We state a central limit theorem that gives convergence of the distribution of the points $f_\omega^n(x)$, after an appropriate scaling. The proof is essentially contained in [32], where a central limit theorem for Markov processes on the half-line is stated.

Theorem 2.5.10. *Let $F^+ \in \mathcal{S}$ and assume $L(0) = 0$ and $L(1) > 0$. Let $x \in (0,1)$. Then, for $a > 0$,*

$$\lim_{n\to\infty} \nu^+\left(\left\{\omega \in \Sigma_2^+ \mid f_\omega^n(x) \ge e^{-a\sqrt{n}}\right\}\right) = \int_0^a \frac{2e^{-\xi^2/2}}{\sqrt{2\pi}}\, d\xi.$$

Proof. Take $x_{n+1} = f_{\omega_n}(x_n)$ with $x_0 = x$. Write $y_n = -\ln(x_n) + \ln(x)$, so that $y_0 = 0$ and $y_n \in [0, \infty)$ if $x_n \in (0, x]$. Write $z_n = \max\{0, y_n\}$.

Lemma 2.5.11. *The moments of z_n^2/n satisfy*

$$\lim_{n\to\infty} \int_{\Sigma_2^+} z_n^{2k}/n^k \, d\nu^+ = \frac{(2k)!}{2^k k!}.$$

Proof. It suffices to follow the proof of [32, Lemma 2.1] and [32, Lemma 2.2]. The proofs in [32] use that the process is null, that is, for any compact interval $J \subset \mathbb{R}$, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \nu^+(\{\omega \mid z_i \in J\}) = 0$. This holds by Theorem 2.5.3. □

As in [32, Theorem 2.1], Lemma 2.5.11 implies that z_n^2/n has a limiting distribution as $n \to \infty$ and

$$\lim_{n\to\infty} \nu^+ \left(\left\{ \omega \in \Sigma_2^+ \mid \frac{z_n^2}{n} \le a^2 \right\} \right) = \int_0^a \frac{2e^{-\xi^2/2}}{\sqrt{2\pi}} \, d\xi.$$

We conclude that

$$\lim_{n\to\infty} \nu^+ \left(\left\{ \omega \in \Sigma_2^+ \mid \frac{y_n}{\sqrt{n}} \le a \right\} \right) = \int_0^a \frac{2e^{-\xi^2/2}}{\sqrt{2\pi}} \, d\xi.$$

Plugging in $y_n = -\ln(x_n) + \ln(x)$ gives the statement of the theorem. □

2.6 Random walk with drift

The material in the previous sections treats all possible combinations of signs of $L(0)$ and $L(1)$ except the case where $L(0) \ge 0$ and $L(1) < 0$ (or vice versa). We put the remaining case in the following result.

Theorem 2.6.1. *Consider $F^+ \in \mathcal{S}$ and assume $L(0) \ge 0$ and $L(1) < 0$. Let $x_0 \in (0,1)$. Then for ν^+-almost every $\omega \in \Sigma_2^+$,*

$$\lim_{n\to\infty} f_\omega^n(x_0) = 1.$$

Proof. Take the proof of Theorem 2.5.9 applied to the inverse skew product system F^{-1}. □

Theorem 2.6.1 establishes forward convergence of $f_\omega^n(x_0)$ to 1 under the given assumptions. Consider $F^+ \in \mathcal{S}$ with $L(0) > 0$ and $L(1) < 0$. Then there is also pullback convergence to 1;

$$\lim_{n\to\infty} f_{\sigma^{-n}\omega}^n(x_0) = 1$$

for ν-almost all $\omega \in \Sigma_2$. It follows from the results in Section 2.5, again by going to the inverse skew product system, that such a pullback convergence does not hold in case $L(0) = 0$, $L(1) < 0$. See [13] for considerations on forward versus pullback convergence in an example of random circle dynamics.

2.A Appendix: invariant measures for step skew product systems

An iterated function system defines a Markov process and as such may admit stationary measures. Their relation with invariant measures for the corresponding one-sided skew product system and its natural extension, the two-sided skew product system, is explored in this section. This is classical material, originating from Furstenberg [17]. A general account of the constructions is found in [4]. We provide a simplified discussion taylored to a setting of step skew product systems over shifts. The reader may also consult the exposition in [41, Chapter 5].

Assume the context from Section 2.2. So consider $\Omega = \{1, \ldots, N\}$ and the family of diffeomorphisms $\mathbb{F} = \{f_1, \ldots, f_N\}$ on M. We pick f_i with probability p_i with $0 < p_i < 1$ and $\sum_{i=1}^N p_i = 1$.

We endow Σ_N with the Borel sigma-algebra, denoted by $\mathcal{F}$. Likewise we take the Borel sigma-algebra $\mathcal{F}^+$ on Σ_N^+. Given the probabilities p_i, we take a Bernoulli measure ν on Σ_N which is determined by its values on cylinders;

$$\nu(C^{k_1,\ldots,k_n}_{\omega_1,\ldots,\omega_n}) = \prod_{i=1}^n p_{\omega_i}.$$

Write ν^+ for the Bernoulli measure on Σ_N^+, defined analogously. For different probability vectors $(p_1, \ldots, p_N)$, the corresponding Bernoulli measures are mutually singular.

Denote by $\mathcal{B}$ the Borel sigma-algebra on M. For a measure m on M and any $\mathcal{B}$-measurable set A we denote the push-forward measure of m by $f_i m$ in which

$$f_i m_i(A) = m_i(f_i^{-1}(A)).$$

Definition 2.A.1. *A stationary measure m on M is a probability measure that equals its average push-forward under the iterated function system $IFS(\mathbb{F})$, i.e. it satisfies*

$$m = \sum_{i=1}^N p_i f_i m.$$

Write $\mathcal{P}_M$ for the space of probability measures on M, equipped with the weak star topology. Write $\mathcal{T} : \mathcal{P}_M \to \mathcal{P}_M$ for the map

$$\mathcal{T} m = \sum_{i=1}^N p_i f_i m.$$

A stationary measure is a fixed point of $\mathcal{T}$.

Lemma 2.A.2. *The map $\mathcal{T}$ is continuous. It also depends continuously on the parameters $p_1, \ldots, p_N$ and $f_1, \ldots, f_N$.*

Proof. Recall that a sequence of measures m_n converges in the weak star topology to a measure m precisely if for each open set A, $\liminf_{n\to\infty} m_n(A) \geqslant m(A)$, see e.g. [39, Theorem III.1.1].

If m_n converges to m it follows that for A open,

$$\liminf_{n\to\infty} f_i m_n(A) = \liminf_{n\to\infty} m_n(f_i^{-1}(A)) \geqslant m(f_i^{-1}(A))$$

since $f_i^{-1}(A)$ is open. That is,

$$\liminf_{n\to\infty} f_i m_n(A) \geqslant f_i m(A)$$

and thus $f_i m_n$ converges to $f_i m$. This argument also shows that $\mathcal{T}$ is continuous.

To prove continuous dependence on $f_1, \ldots, f_N$, consider a sequence of maps $f_{i,n}$ converging to f_i. By inner regularity, for an open set $O \subset M$ one has $m(O) = \sup_{C \subset O} m(C)$, where C runs over compact subsets of O. So also, given $\varepsilon > 0$, for $A \subset M$ open, there exists compact $K \subset A$ with $m(f_i^{-1}(K)) \geqslant m(f_i^{-1}(A)) - \varepsilon$.

Then, for A open, given $\varepsilon > 0$ there are $n_0 > 0$ and $K \subset A$ so that $f_i^{-1}(K) \subset f_{i,n}^{-1}(A)$ for $n \geqslant n_0$ and $m(f_i^{-1}(K)) \geqslant m(f_i^{-1}(A)) - \varepsilon$. So,

$$\begin{aligned}
\liminf_{n\to\infty} f_{i,n} m(A) &= \liminf_{n\to\infty} m(f_{i,n}^{-1}(A)) \\
&\geq \lim_{n_0\to\infty} m\left(\bigcap_{n\geq n_0} f_{i,n}^{-1}(A)\right) \\
&\geq m(f_i^{-1}(K)) \\
&\geq m(f_i^{-1}(A)) - \varepsilon.
\end{aligned}$$

As this holds for any ε, we get

$$\liminf_{n\to\infty} f_{i,n} m(A) \geqslant m(f_i^{-1}(A))$$

and hence that $f_{i,n} m$ converges to $f_i m$. This argument shows that $\mathcal{T}$ depends continuously on $f_1, \ldots, f_N$, continuous dependence on parameters $p_1, \ldots, p_N$ is clear. $\square$

The same type of argument shows that the map $\mathcal{T}_\varepsilon$, appearing in the proof of Theorem 2.4.1, is continuous and changes continuously with ε. The set of fixed points of $\mathcal{T}$ changes upper semi-continuously in the Hausdorff metric if parameters $p_1, \ldots, p_N$ and $f_1, \ldots, f_N$ are varied. So if m is a unique fixed point for $\mathcal{T}$, $\mathcal{T}_\varepsilon \to \mathcal{T}$ as $\varepsilon \to 0$ and $\mathcal{T}_\varepsilon m_\varepsilon = m_\varepsilon$, then $m_\varepsilon \to m$ as $\varepsilon \to 0$.

Lemma 2.A.3. *A probability measure m is a stationary measure if and only if $\mu^+ = \nu^+ \times m$ is an invariant measure of F^+ with marginal ν^+ on Σ_N^+.*

Proof. Consider the following calculation for product sets $C \times B \subset \Sigma_N^+ \times M$ of a cylinder $C = C_{i_0,\dots,i_{n-1}}^{0,\dots,n-1}$ and a Borel set B:

$$\begin{aligned} F^+(\nu^+ \times m)(C \times B) &= \nu^+ \times m((F^+)^{-1}(C \times B)) \\ &= \sum_{i=1}^N \nu^+ \times m\left(C_{i,i_0,\dots,i_{n-1}}^{0,1,\dots,n} \times f_i^{-1}(B)\right) \\ &= \sum_{i=1}^N p_i \nu^+(C) m(f_i^{-1}(B)) \\ &= \sum_{i=1}^N p_i \nu^+(C) f_i m(B). \end{aligned}$$

If m is a stationary measure, then the last expression equals $\nu^+(C)m(B) = \nu^+ \times m(C \times B)$, so that $F^+(\nu^+ \times m)(C \times B) = \nu^+ \times m(C \times B)$. Since the product sets generate the σ-algebra, this proves F^+-invariance of $\nu^+ \times m$. Similarly, if $\nu^+ \times m$ is F^+-invariant, then the last expression equals $\nu^+ \times m(C \times B) = \nu^+(C)m(B)$ and this proves $\sum_{i=1}^N p_i f_i m(B) = m(B)$. □

Let m be a stationary measure for M. We say that m is ergodic if $\nu^+ \times m$ is ergodic for F^+. A point (ω, x) is said to be a generic point for an ergodic measure $\nu^+ \times m$, if the orbit is distributed according to the measure.

Write $\pi : \Sigma_2 \to \Sigma_2^+$ for the natural projection $(\omega_n)_{-\infty}^\infty \mapsto (\omega_n)_0^\infty$. The Borel sigma-algebra $\mathcal{F}^+$ on Σ_N^+ yields a sigma-algebra $\mathcal{F}_0 = \pi^{-1}\mathcal{F}^+$ on Σ_N. A measure μ on $\Sigma_N \times M$ with marginal ν has conditional measures μ_ω on the fibers $\{\omega\} \times M$, such that

$$\mu(A) = \int_{\Sigma_N} \mu_\omega(A_\omega)\, d\nu(\omega) \tag{2.A.1}$$

for measurable sets A, where we have written

$$A_\omega = A \cap (\{\omega\} \times M).$$

A measure μ^+ on $\Sigma_N^+ \times M$ with marginal ν^+ likewise has conditional measures μ_ω^+. It is convenient to consider ν^+ as a measure on Σ_N with sigma-algebra $\mathcal{F}_0$ and μ^+ as a measure on $\Sigma_N \times M$ with sigma-algebra $\mathcal{F}_0 \otimes \mathcal{B}$. When $\omega \in \Sigma_N$ we will write μ_ω^+ for the conditional measures $\mu_{\pi\omega}^+$. The spaces of measures are equipped with the weak star topology.

Invariant measures for F^+ with marginal ν^+ correspond to invariant measures for F with marginal ν in a one-to-one relationship, as detailed in Proposition 2.A.4 below. This is a special case of [4, Theorem 1.7.2]. The result implies that stationary measures correspond one-to-one to specific invariant measures for F with marginal ν.

Write $\Sigma_N = \Sigma_N^- \times \Sigma_N^+$, where Σ_N^- consists of the past parts $(\omega_i)_{-\infty}^{-1}$ of sequences ω. We have a natural projection

$$\Pi : \Sigma_N^- \times \Sigma_N^+ \times M \to \Sigma_N^+ \times M.$$

Proposition 2.A.4. *Let μ^+ be an F^+-invariant probability measure with marginal ν^+. Then, there exists an F-invariant probability measure μ with marginal ν and conditional measures*

$$\mu_\omega = \lim_{n\to\infty} f^n_{\sigma^{-n}\omega}\mu^+_{\sigma^{-n}\omega}, \tag{2.A.2}$$

ν-almost surely.

Let μ be an F-invariant probability measure with marginal ν. Then,

$$\mu^+ = \Pi\mu \tag{2.A.3}$$

is an F^+-invariant probability measure with marginal ν^+.

The correspondence $\mu \leftrightarrow \mu^+$ given by (2.A.2), (2.A.3) *is one-to-one. Furthermore, through these relations, F^+-invariant product measures $\mu^+ = \nu^+ \times m$ correspond one-to-one with F-invariant product measures $\mu = \nu^+ \times \vartheta$ on $\Sigma^+_N \times (\Sigma^-_N \times M)$.*

Remark 2.A.5. *Consider μ^+ as a measure on $\Sigma_N \times M$ with sigma-algebra $\mathcal{F}_0 \otimes \mathcal{B}$. Observe that $F^n(\mu^+)$ has conditional measures $f^n_{\sigma^{-n}\omega}\mu^+_{\sigma^{-n}\omega}$ on $\{\omega\} \times M$. Hence,* (2.A.2) *reads*

$$\mu = \lim_{n\to\infty} F^n(\mu^+).$$

Remark 2.A.6. *From the characterization of ergodic probability measures as extremal points in the set of invariant probability measures, the one-to-one correspondence $\mu \leftrightarrow \mu^+$ implies that μ is ergodic if and only if μ^+ is ergodic.*

Proof of Proposition 2.A.4. Note that $\mathcal{F}_s = \sigma^s\mathcal{F}_0$ are sigma-algebras on Σ_N with $\mathcal{F}_s \uparrow \mathcal{F}$. For a Borel set $B \subset M$, define

$$\upsilon_t(\omega) = f^t_{\sigma^{-t}\omega}\mu^+_{\sigma^{-t}\omega}(B).$$

Calculate, for $A_s = \sigma^s A_0 \in \mathcal{F}_s$ and $0 \le s \le t$,

$$\begin{aligned}
\int_{A_s} \upsilon_t(\omega)\, d\nu(\omega) &\overset{(1)}{=} \int_{A_s} f^t_{\sigma^{-t}\omega}\mu^+_{\sigma^{-t}\omega}(B)\, d\nu(\omega)\\
&\overset{(2)}{=} \int_{A_0} f^t_{\sigma^{s-t}\omega}\mu^+_{\sigma^{s-t}\omega}(B)\, d\nu(\omega)\\
&\overset{(3)}{=} \int_{A_0} f^s_{\omega}\mu^+_{\omega}(B)\, d\nu(\omega)\\
&\overset{(4)}{=} \int_{A_s} f^s_{\sigma^{-s}\omega}\mu^+_{\sigma^{-s}\omega}(B)\, d\nu(\omega)\\
&\overset{(5)}{=} \int_{A_s} \upsilon_s(\omega)\, d\nu(\omega).
\end{aligned}$$

Here (1) and (5) is the definition of υ_t, (2) and (4) are by σ-invariance of ν, and (3) is by F^+-invariance of μ^+ (see Lemma 2.A.7 and Corollary 2.A.8 below for a derivation).

The above calculation shows that υ_t is a martingale with respect to the filtration $\mathcal{F}_t$. Hence the limit $\lim_{t\to\infty} \upsilon_t(\omega)$ exists. By the Vitali-Hahn-Saks theorem, see [16, Theorem III.10], the limit for varying Borel sets B defines a measure, μ_ω. To obtain that the resulting measure μ is F-invariant, we refer to Remark 2.A.5. Since F acts continuous on the space of probability measures, the limit $\lim_{n\to\infty} F^n(\mu^+)$ is F-invariant.

It remains to show that μ and μ^+ are in one-to-one correspondence. We wish to show that, given μ and computing $\mu^+ = \Pi\mu$, the formula (2.A.2) recovers μ. Note, again with $A_t = \sigma^t A_0 \in \mathcal{F}_t$ for $0 \le t$,

$$\begin{aligned}
\int_{A_t} \upsilon_t(\omega)\, d\nu(\omega) &= \int_{A_t} f^t_{\sigma^{-t}\omega}\mu^+_{\sigma^{-t}\omega}(B)\, d\nu(\omega)\\
&= \int_{A_0} f^t_\omega \mu^+_\omega(B)\, d\nu(\omega)\\
&= \int_{A_0} \mu^+_\omega((f^t_\omega)^{-1}(B))\, d\nu(\omega)\\
&= \mu^+\left(\bigcup_{\omega\in A_0} \{\omega\}\times (f^t_\omega)^{-1}(B)\right)\\
&= \mu\left(\bigcup_{\omega\in A_0} \{\omega\}\times (f^t_\omega)^{-1}(B)\right)\\
&= \mu\left((F^t)^{-1}(A_t\times B)\right)\\
&= \int_{A_t} \mu_\omega(B)\, d\nu(\omega).
\end{aligned}$$

As $\mathcal{F}_t \uparrow \mathcal{F}$, this shows that υ_t converges to μ as $t\to\infty$.

If μ is a product measure $\nu^+\times\vartheta$ on $\Sigma^+_N\times(\Sigma^-_N\times M)$, then clearly $\mu^+ = \Pi\mu$ is a product measure on $\Sigma^+_N\times M$. In the other direction, if $\mu^+ = \nu^+\times m$, then (2.A.2) reads

$$\mu_\omega = \lim_{n\to\infty} f^n_{\sigma^{-n}\omega} m,$$

so that μ_ω does not depend on the future $\omega^+ = (\omega_n)_0^\infty$ of ω. For a product set $A = C^+\times B\subset \Sigma^+_N\times(\Sigma^-_N\times M)$, (2.A.1) yields

$$\mu(A) = \int_{\Sigma^+_N}\int_{\Sigma^-_N} \mu_\omega(A_\omega)\, d\nu^-(\omega^-)d\nu^+(\omega^+).$$

Since μ_ω depends on the past ω^- alone, this can be written as $\mu(A) = \nu^+\times\vartheta(C^+\times B) = \nu^+(C^+)\vartheta(B)$. So μ is a product measure $\nu^+\times\vartheta$ on $\Sigma^+_N\times(\Sigma^-_N\times M)$. $\square$

The following lemma draws conclusions from F^+-invariance of μ^+.

Lemma 2.A.7. *For $A_0\in\mathcal{F}^+$, $B\in\mathcal{B}$, $0\le s\le t$,*

$$\int_{\sigma^{s-t}A_0} f^t_\omega\mu^+_\omega(B)\, d\nu^+(\omega) = \int_{\sigma^{s-t}A_0} f^s_{\sigma^{t-s}\omega}\mu^+_{\sigma^{t-s}\omega}(B)\, d\nu^+(\omega).$$

Proof. Write $A_{s-t} = \sigma^{s-t} A_0$ and compute, using F^+-invariance of μ^+,

$$\begin{aligned}
\int_{A_{s-t}} f^t_\omega \mu^+_\omega(B)\, d\nu^+(\omega) &= \int_{A_{s-t}} f^s_{\sigma^{t-s}\omega} f^{t-s}_\omega \mu^+_\omega(B)\, d\nu^+(\omega) \\
&= \int_{A_{s-t}} \mu^+_\omega((f^{t-s}_\omega)^{-1}(f^s_{\sigma^{t-s}\omega})^{-1}(B))\, d\nu^+(\omega) \\
&= \mu^+\left(\bigcup_{\omega \in A_{s-t}} \{\omega\} \times (f^{t-s}_\omega)^{-1}(f^s_{\sigma^{t-s}\omega})^{-1}(B) \right) \\
&= \mu^+\left(\bigcup_{\omega \in A_0} \{\sigma^{s-t}\omega\} \times (f^{t-s}_{\sigma^{s-t}\omega})^{-1}(f^s_\omega)^{-1}(B) \right) \\
&= \mu^+\left((F^+)^{s-t} \bigcup_{\omega \in A_0} \{\omega\} \times (f^s_\omega)^{-1}(B) \right) \\
&= \mu^+\left(\bigcup_{\omega \in A_0} \{\omega\} \times (f^s_\omega)^{-1}(B) \right) \\
&= \int_{A_0} \mu^+_\omega((f^s_\omega)^{-1}(B))\, d\nu^+(\omega) \\
&= \int_{A_0} f^s_\omega \mu^+_\omega(B)\, d\nu^+(\omega).
\end{aligned}$$

As $\int_{A_0} f^s_\omega \mu^+_\omega(B)\, d\nu^+(\omega) = \int_{A_{s-t}} f^s_{\sigma^{t-s}\omega} \mu^+_{\sigma^{t-s}\omega}(B)\, d\nu^+(\omega)$ by σ-invariance of ν^+, this concludes the argument. □

Corollary 2.A.8. *The lemma implies that for $A_0 \in \mathcal{F}_0$, $B \in \mathcal{B}$, for $0 \le s \le t$,*

$$\int_{A_0} f^t_{\sigma^{s-t}\omega} \mu^+_{\sigma^{s-t}\omega}(B)\, d\nu(\omega) = \int_{A_0} f^s_\omega \mu^+_\omega(B)\, d\nu(\omega).$$

Note that for the natural extension, F-invariance of μ means

$$f^t_{\sigma^{s-t}\omega} \mu_{\sigma^{s-t}\omega} = f^s_\omega \mu_\omega$$

for $0 \le s \le t$ and for ν-almost all $\omega \in \Sigma_N$.

Bibliography

[1] J. C. Alexander, J. A. Yorke, Z. You, I. Kan. Riddled basins. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 2:795–813, 1992.

[2] L. Alsedà, M. Misiurewicz. Random interval homeomorphisms. *Publ. Mat.* 58:15–36, 2014.

[3] V. A. Antonov. Modeling cyclic evolution processes: Synchronization by means of a random signal. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* 2:67–76, 1984.

[4] L. Arnold. *Random dynamical systems.* Springer Verlag, 1998.

[5] K. B. Athreya, J. Dai. Random logistic maps I. *Journal of Theoretical Probability* 13:595–608, 2000.

[6] K. B. Athreya, H. J. Schuh. Random logistic maps II. The Critical case. *Journal of Theoretical Probability* 16:813–830, 2003.

[7] L. Barreira, Y. Pesin. *Nonuniform hyperbolicity. Dynamics of systems with nonzero Lyapunov exponents.* Cambridge University Press, 2007.

[8] P. H. Baxendale. Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms. *Probability Theory and Related Fields* 81:521–554, 1989.

[9] P. Bergé, Y. Pomeau, C. Vidal. *Order within chaos. Towards a deterministic approach to turbulence.* John Wiley & Sons Ltd., 1986.

[10] C. Bonatti, L. J. Díaz, M. Viana. *Dynamics beyond uniform hyperbolicity.* Springer-Verlag, 2005.

[11] A. Bonifant, J. Milnor. Schwarzian Derivatives and Cylinder Maps. *Fields Institute Communications* 53:1–21, 2008.

[12] K. L. Chung. *A course in probability theory.* Harcourt, Brace & World, Inc. 1968.

[13] H. Crauel. A uniformly exponential random forward attractor which is not a pullback attractor. *Arch. Math.*, 78:329–336, 2002.

[14] B. Deroin, V. A. Kleptsyn, A. Navas. Sur la dynamique unidimensionnelle en régularité intermédiaire. *Acta Math.* 199:199–262, 2007.

[15] M. D. Donsker, S. R. S. Varadhan. Asymptotic evaluation of certain Markov process expectations for large time. I. *Comm. Pure Appl. Math.* 28:1–47, 1975.

[16] J. L. Doob. *Measure theory.* Springer Verlag, 1993.

[17] H. Furstenberg. Boundary theory and stochastic processes on homogeneous spaces. In *Proc. Sympos. Pure Math.* Volume 26, pages 193–229, 1973.

[18] M. Gharaei, A.J. Homburg. Skew products of interval maps over subshifts. J. Difference Equ. Appl. 22(7), 941-958, 2016.

[19] A. S. Gorodetskiĭ, Yu. S. Il′yashenko. Certain new robust properties of invariant sets and attractors of dynamical systems. *Funct. Anal. Appl.*, 33:95–105, 1999.

[20] A. S. Gorodetskiĭ, Yu. S. Il′yashenko, V. A. Kleptsyn, and M. B. Nal′skiĭ. Nonremovability of zero Lyapunov exponents. *Funct. Anal. Appl.*, 39:21–30, 2005.

[21] J. F. Heagy, N. Platt, S. M. Hammel. Characterization of on-off intermittency *Phys. Rev. E.* 49:1140–1150, 1994.

[22] Yu. S. Il′yashenko. Thick attractors of step skew products. *Regular and chaotic dynamics* 15:328–334, 2010.

[23] Yu. S. Il′yashenko. Thick attractors of boundary preserving diffeomorphisms. *Indag. Math. (N.S.)* 22:257–314, 2011.

[24] Yu. S. Il′yashenko, V. A. Kleptsyn, P. Saltykov. Openness of the set of boundary preserving maps of an annulus with intermingled attracting basins. *J. fixed point theory appl.* 3:449–463, 2008.

[25] I. Kan. Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin. *Bull. Amer. Math. Soc.* 31:68–74, 1994.

[26] G. Keller. Stability index for chaotically driven concave maps. *J. London Math. Soc.* 89:603–622, 2014.

[27] V. A. Kleptsyn, M. B. Nal′skiĭ. Contraction of orbits in random dynamical systems on the circle. *Funct. Anal. Appl.* 38:267–282, 2004.

[28] V. A. Kleptsyn, P. S. Saltykov. On C^2-stable effects of intermingled basins of attractors in classes of boundary-preserving maps. *Trans. Moscow Math. Soc.* 72:193–217, 2011.

[29] V. A. Kleptsyn, D. Volk. Physical measures for nonlinear random walks on interval. *Mosc. Math. J.* 14:339–365, 2014.

[30] P. E. Kloeden, M. Rasmussen. *Nonautonomous dynamical systems.* Amer. Math. Soc., 2011.

[31] J. Lamperti. Criteria for the recurrence or transience of stochastic processes, I. *J. Math. Anal. and Appl.* 1:314–330, 1960.

[32] J. Lamperti. A new class of probability limit theorems. *J. Math. and Mech.* 11:749–772, 1962.

[33] J. Lamperti. Criteria for stochastic processes II: passage-time moments *J. Math. Anal. and Appl.* 7:127–145, 1963.

[34] J. Milnor. On the concept of attractor. *Commun. Math. Phys.*, 99:177–195, 1985.

[35] E. Ott, J. Sommerer, J. Alexander, I. Kan, J. Yorke. Scaling behavior of chaotic systems with riddled basins. *Physical Review Letters* 71:4134–4137, 1993.

[36] L. M. Pecora, T. L. Carroll, Synchronization in chaotic systems. *Phys. Rev. Lett.* 64:821–824, 1990.

[37] A. Pikovsky, M. Rosenblum, J. Kurths. *Synchronization. A universal concept in nonlinear sciences.* Cambridge University Press, 2001.

[38] N. Platt, E. A. Spiegel, C. Tresser. On-off intermittency: a mechanism for bursting. *Phys. Rev. Letters* 70:279–282, 1993.

[39] A. N. Shiryayev. *Probability.* Springer Verlag, 1984.

[40] J. Stark. Invariant graphs for forced systems. *Phys. D* 109:163–179, 1997.

[41] M. Viana. *Lectures on Lyapunov exponents.* Cambridge University Press, 2014.

[42] H. Zmarrou, A. J. Homburg. Bifurcations of stationary densities of random diffeomorphisms. *Ergod. Th. Dyn. Systems* 27:1651–1692, 2007.

[43] H. Zmarrou, A. J. Homburg. Dynamics and bifurcations of random circle diffeomorphisms. *Discrete Contin. Dyn. Syst. Ser. B*, 10:719–731, 2008.

Chapter 3

Iterated function systems of logistic maps: synchronization and intermittency

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Abstract

We discuss iterated function systems generated by finitely many logistic maps, with a focus on synchronization and intermittency. We provide sufficient conditions for synchronization, involving negative Lyapunov exponents and minimal dynamics. A number of results that clarify the scope of these conditions are included. We analyze a mechanism for intermittency that involves the full map $x \mapsto 4x(1-x)$ as one of the generators of the iterated function system. For iterated function systems generated by $x \mapsto 2x(1-x)$ and $x \mapsto 4x(1-x)$ we prove the existence of a σ-finite stationary measure.

3.1 Introduction

Iterated function systems are given by a (finite) collection of continuous maps on a metric space, that are composed for iterations. The maps are typically picked at random each iterate. They have been studied extensively because of their role in the study of fractals [17, 10]. The dynamics of iterated function systems can be studied using a description as a skew product system over a shift operator. As such they provide case studies for nonuniformly hyperbolic dynamics, which is another reason for their study. Indeed, phenomena that have been observed in skew product systems coming from iterated function systems frequently have analogues in more general skew product systems and more general dynamical systems (see [14] for an example where this line of thought is made explicit).

We take iterated function systems given by a collection of $k > 1$ logistic maps $f_i : I \to I$, $1 \leqslant i \leqslant k$, with $I = [0, 1]$ and

$$f_i(x) = \rho_i x(1 - x),$$

$0 < \rho_i \leqslant 4$. The dynamics of logistic maps is a paradigm for chaotic dynamics, see [22], making it interesting to consider iterated function systems of logistic maps.

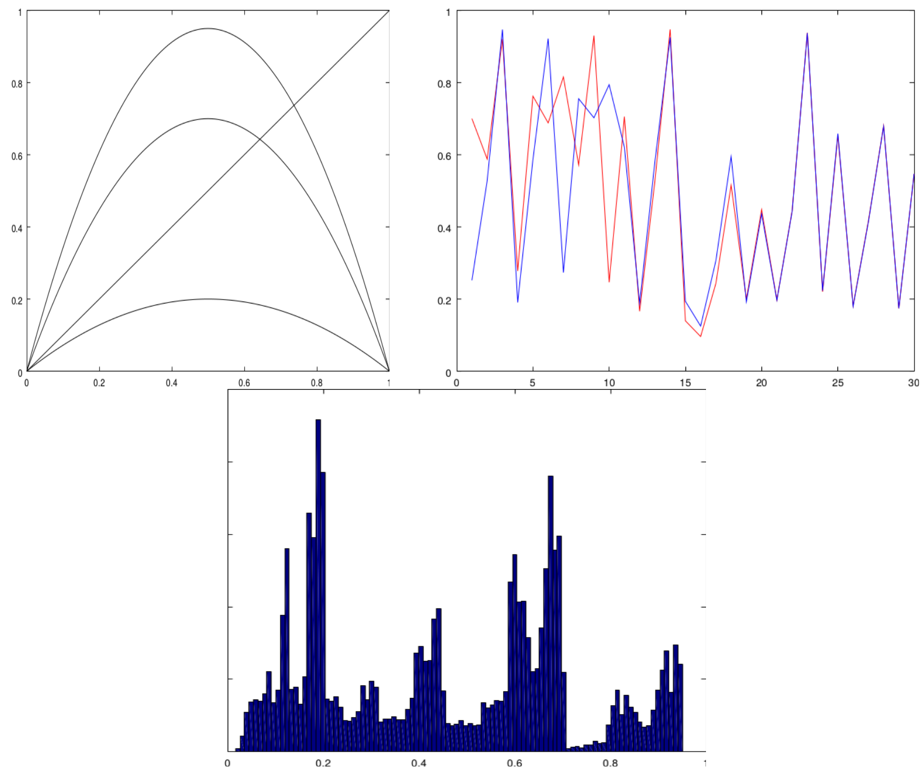

Figure 3.1: *This figure illustrates the phenomenon of synchronization. Considered is the iterated function system generated by* $f_1(x) = 0.8x(1-x)$ *(chosen with probability* $p_1 = 1/4$*),* $f_2(x) = 2.8x(1-x)$ *(chosen with probability* $p_2 = 1/2$*),* $f_2(x) = 3.8x(1-x)$ *(chosen with probability* $p_3 = 1/4$*). Graphs are drawn in the first frame. The second frame shows numerically computed time series for two different initial conditions, where the points are connected by lines. The two orbits appear to converge to each other. The third frame shows the histogram of a numerically computed orbit, indicating that orbits occupy an entire interval.*

We focus on dynamics which can be viewed as simple dynamics in the skew product system setting, for instance characterized by negative Lyapunov exponents. We start with an investigation of

> *synchronization: orbits of different initial conditions converge to each other under identical compositions of logistic maps.*

Figure 3.1 illustrates it by showing two orbit pieces of an iterated function system with a visible convergence of orbits. The figure contains also the histogram of a numerically computed orbit, illustrating that even though different orbits converge to each other, the orbits themselves occupy large parts of the interval. Steinsaltz [28] wrote a fundamental paper on contractive dynamics for random iteration of logistic maps, with noise such that an irreducibility condition on the corresponding Markov chain holds. We continue his investigation in a context of iterated function systems generated by finitely many logistic maps. We use a dynamical systems theory approach, adopting a skew product systems point of view. We prove a theorem on synchronization in iterated function systems of logistic maps under some assumptions, see Theorem 3.3.1 and Theorem 3.3.15. One of the assumptions is that the fixed point at 0 is repelling on average, so that typical orbits will not converge to 0. Other assumptions are on negative Lyapunov exponents and on minimality of the dynamics of the iterated function system. Precise statements and a detailed discussion are in Section 3.3.

If the fixed point at 0 is neutral on average, Athreya and Schuh [4] prove the occurrence of

> *intermittency: for typical orbits the set of iterates for which the orbit is near* 0*, has full density, but orbits do not stay near* 0.

The left frame of Figure 3.2 illustrates a time series. This kind of intermittency has been called on-off intermittency [25, 15]. We discuss a different mechanism for intermittency, where the iterated function system contains both the map $x \mapsto 2x(1-x)$, for which the critical point is a superstable fixed point, and the map $x \mapsto 4x(1-x)$, for which the critical point is mapped onto 0 in two iterates. We assume the fixed point at 0 to be repelling on average. In our discussion of synchronization the possibility of iterating the map $x \mapsto 4x(1-x)$ is ruled out. That inclusion of this map can give rise to new phenomena was earlier observed by Högnäs and Carlsson [8]. An intermittent time series under this mechanism is illustrated in the right frame of Figure 3.2. Precise statements and proofs of intermittency are in Section 3.4, see in particular Theorem 3.4.1 and Theorem 3.4.3. The first of these two theorems establishes intermittency by studying time series. The second theorem gives more precise information through σ-finite stationary measures. Section 3.4.3 includes a brief discussion of other examples of intermittency, such as involving superstable periodic orbits. Figure 3.3 pictures two examples.

For a more precise discussion of our results, we start with some generalities to introduce concepts and notation. Then we present our main results on synchronization in Section 3.3 and intermittency in Section 3.4.

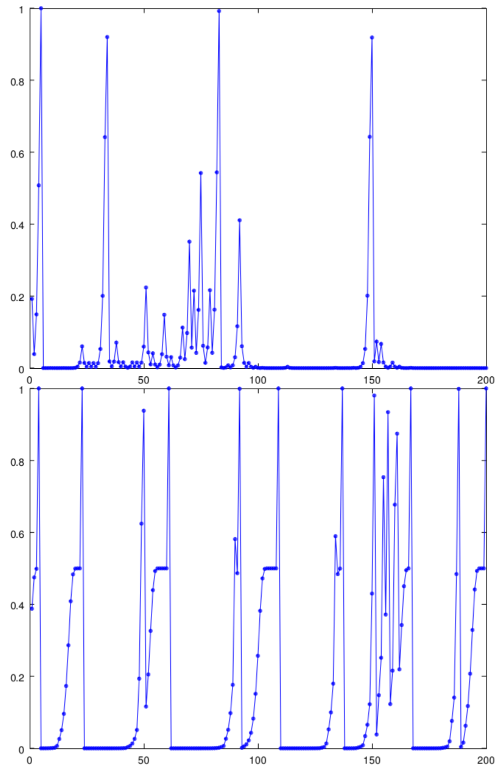

Figure 3.2: *Two examples of intermittent time series. The left frame shows numerically computed time series for the iterated function system generated by* $f_1(x) = \frac{1}{4}x(1-x)$ *(chosen with probability* $p_1 = 0.5$*) and* $f_2(x) = 4x(1-x)$ *(chosen with probability* $p_2 = 0.5$*). This iterated function system has a vanishing Lyapunov exponent at* 0*. The right frame shows a numerically computed time series for the iterated function system generated by* $f_1(x) = 4x(1-x)$ *(chosen with probability* $p_1 = 0.3$*) and* $f_2(x) = 2x(1-x)$ *(chosen with probability* $p_2 = 0.7$*). Here* 0 *is repelling on average, but nonetheless typical orbits have full density of their iterates near* 0*.*

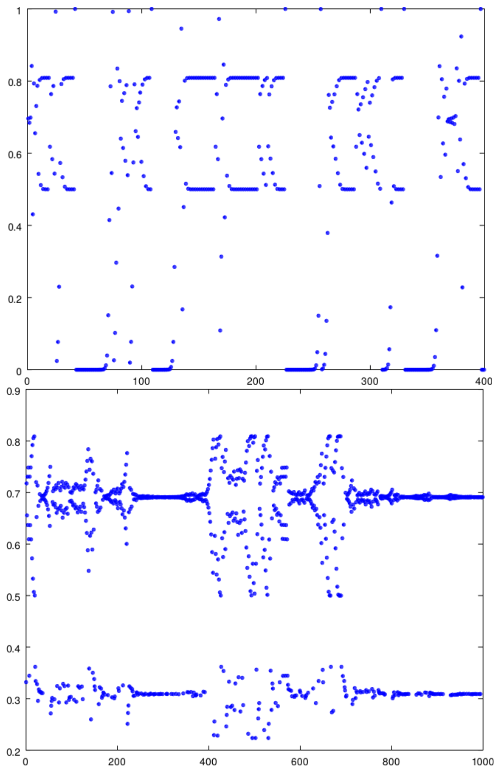

Figure 3.3: *The left frame illustrates Theorem 3.4.11: numerically computed time series for the iterated function system generated by* $f_1(x) = (1+\sqrt{5})x(1-x)$ *(chosen with probability* $p_1 = 0.85$*) and* $f_2(x) = 4x(1-x)$ *(chosen with probability* $p_2 = 0.15$*). The map* f_1 *possesses a superstable period two orbit. The right frame illustrates Theorem 3.4.13: numerically computed time series for the iterated function system generated by* $f_1(x) = (1+\sqrt{5})x(1-x)$ *and* $f_2(x) = (1+\frac{\sqrt{5}}{5})x(1-x)$ *chosen with probabilities that yield zero Lyapunov exponents on the invariant set consisting of the two positive fixed points of* f_1 *and* f_2*.*

3.2 Generalities

Throughout the paper, $f_1, \ldots, f_k$ will stand for logistic maps

$$f_i(x) = \rho_i x(1-x),\ 0 < \rho_i \leqslant 4,$$

on $I = [0,1]$. The study in this paper makes use of descriptions with skew product systems and properties of their invariant measures. The necessary material is introduced in this section. This fixes also notation for the rest of the paper.

3.2.1 Iterated function systems

Denoting $\mathbb{F} = \{f_1, \ldots, f_k\}$, the iterated function system IFS$(\mathbb{F})$ is the action on I of the semi-group generated by $f_1, \ldots, f_k$. A set $A \subset I$ is called invariant for the iterated function system IFS$(\mathbb{F})$ if $\mathbb{F}(A) = A$, where $\mathbb{F}(A) = \cup_{i=1}^k f_i(A)$. A sequence $\{x_n : n \in \mathbb{N}\}$ is called a branch of an orbit of IFS$(\mathbb{F})$ if for each $n \in \mathbb{N}$ there is $f_n \in \mathbb{F}$ such that $x_{n+1} = f_n(x_n)$. The iterated function system IFS$(\mathbb{F})$ is minimal on an invariant set A if any orbit of $x \in A$ under IFS$(\mathbb{F})$ has a branch which is dense in A.

The logistic maps $f_i(x) = \rho_i x(1-x)$ are chosen independently from a fixed distribution; f_i is picked with probability p_i with $0 < p_i < 1$ and $\sum_{i=1}^k p_i = 1$. As this defines a Markov process, a central role is played by stationary measures: a stationary measure m is a probability measure on I that satisfies

$$m = \sum_{i=1}^k p_i f_i m.$$

That is, a stationary measure is equal to its averaged push forward under the maps f_i, where the pushforward $f_i m$ is defined by $f_i m(A) = m(f_i^{-1}(A))$ for Borel sets $A \subset I$. The support $\operatorname{supp}(m)$ of a stationary measure m is the complement of the largest open set A for which $m(A) = 0$.

Let $\mathcal{M}_I$ be the space of all Borel probability measures on I endowed with the weak-star topology. The topological space $\mathcal{M}_I$ is metrizable, we will take a metric $d_{\mathcal{M}_I}$ on $\mathcal{M}_I$ that generates the weak star topology, see e.g. [21].

Define the transformation $\mathcal{T} : \mathcal{M}_I \to \mathcal{M}_I$ by

$$\mathcal{T}m = \sum_{i=1}^k p_i f_i m. \tag{3.2.1}$$

Note that the fixed points of $\mathcal{T}$ are precisely the stationary measures of the iterated function system.

Lemma 3.2.1. *The map $\mathcal{T}$ is continuous. It also depends continuously on the parameters $p_1, \ldots, p_k$ and $\rho_1, \ldots, \rho_k$.*

Proof. The proof of [13, Lemma A.1] applies. □

3.2.2 Skew product systems

Associated to an iterated function system is a skew product system over the shift on a symbol space. Let $\Sigma_k^+ = \{1, \ldots, k\}^{\mathbb{N}}$ and $F^+ : \Sigma_k^+ \times I \to \Sigma_k^+ \times I$ be given by

$$F^+(\omega, x) = (\sigma^n \omega, f_{\omega_0}(x)).$$

Here σ is the left shift operator acting on a sequence $\omega = (\omega_i)_{i \in \mathbb{N}}$ by $(\sigma\omega)_i = \omega_{i+1}$. For notational convenience we write

$$\left(F^+\right)^n(\omega, x) = (\sigma^n \omega, f_\omega^n(x))$$

for $n \geqslant 1$, so

$$f_\omega^n = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}.$$

We will adopt the notation f_ω^n also if only $\omega_0, \ldots, \omega_{n-1}$ are given.

Given probabilities p_i, one has the product measure, or Bernoulli measure, ν^+ on Σ_k^+. If $C^{0,\ldots,n-1}_{i_0,\ldots,i_{n-1}} \subset \Sigma_k^+$ is the cylinder

$$C^{0,\ldots,n-1}_{i_0,\ldots,i_{n-1}} = \{\omega \in \Sigma_k^+ \; ; \; \omega_j = i_j, 0 \leqslant j \leqslant n-1\},$$

then

$$\nu^+(C^{0,\ldots,n-1}_{i_0,\ldots,i_{n-1}}) = p_{i_0}^{\ell_0} \cdots p_{i_{n-1}}^{\ell_{n-1}}$$

if ℓ_i is the number of symbols i in $i_0, \ldots, i_{n-1}$.

A direct computation gives the following well known correspondence between stationary measures and invariant measures for F^+ with marginal ν^+.

Lemma 3.2.2. *A probability measure m is a stationary measure if and only if $\mu^+ = \nu^+ \times m$ is an invariant measure of F^+ with marginal ν^+ on Σ_k^+.*

Proof. See e.g. [13, Lemma A.2]. □

A stationary measure m is called ergodic if $\nu^+ \times m$ is ergodic for F^+. As a consequence of Birkhoff's ergodic theorem, see e.g. [11, Corollary 4.20], for an ergodic stationary measure m we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f_\omega^i(x)} = m, \tag{3.2.2}$$

with convergence in the weak star topology, for $\nu^+ \times m$-almost all (ω, x). So we recover m from the distribution of points in typical orbits. A point (ω, x) is said to be a generic point for an ergodic measure $\nu^+ \times m$, if the orbit is distributed according to the measure.

Similar to F^+, an associated skew product system over the two sided shift is given: with $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$, $F : \Sigma_k \times I \to \Sigma_k \times I$ is defined by

$$F(\omega, x) = (\sigma\omega, f_\omega(x)).$$

The map F on $\Sigma_k \times I$ is an extension of F^+ on $\Sigma_k^+ \times I$. In the set-up of invertible fiber maps f_ω, F is invertible (in such a set-up F is the natural extension). In our case F is not invertible, because the fiber maps are not invertible. Write ν for the product measure on Σ_k corresponding to the probabilities p_i.

A key role in our study is played by ergodic invariant measures for the skew product systems. The relation between invariant measures with marginal ν^+ for F^+ and invariant measures with marginal ν for F is discussed in [13]. The material comes from standard sources such as [1]. A difference with the material in [1, 13] is that here the fiber maps f_ω are not invertible and so F is an extension of F^+ but not the natural extension. The following material is usually developed for natural extensions, as in [13], but applies to the current setting as well.

Denote by $\mathbf{B}$ the Borel sigma-algebra on I. Write $\pi : \Sigma_k \to \Sigma_k^+$ for the natural projection

$$\pi(\omega_n)_{-\infty}^{\infty} = (\omega_n)_0^{\infty}.$$

The Borel sigma-algebra $\mathbf{F}^+$ on Σ_k^+ yields a sigma-algebra $\mathbf{F}_0 = \pi^{-1}\mathbf{F}^+$ on Σ_k. A measure μ on $\Sigma_k \times I$ with marginal ν has conditional measures μ_ω on the fibers $\{\omega\} \times I$, such that

$$\mu(A) = \int_{\Sigma_k} \mu_\omega(A_\omega)\, d\nu(\omega)$$

for measurable sets A, where we have written

$$A_\omega = A \cap (\{\omega\} \times I).$$

A measure μ^+ on $\Sigma_k^+ \times I$ with marginal ν^+ likewise has conditional measures μ_ω^+. It is convenient to consider ν^+ as a measure on Σ_k with sigma-algebra $\mathbf{F}_0$ and μ^+ as a measure on $\Sigma_k \times I$ with sigma-algebra $\mathbf{F}_0 \otimes \mathbf{B}$. When $\omega \in \Sigma_k$ we will write μ_ω^+ for the conditional measures $\mu_{\pi\omega}^+$. The spaces of measures are equipped with the weak star topology.

Invariant measures for F^+ with marginal ν^+ correspond to invariant measures for F with marginal ν in a one-to-one relationship, as detailed in Proposition 3.2.3 below. This is a special case of [1, Theorem 1.7.2], see [13, Proposition A.1]. The result implies that stationary measures correspond one-to-one to specific invariant measures for F with marginal ν.

Write $\Sigma_k = \Sigma_k^- \times \Sigma_k^+$, where Σ_k^- consists of the past parts $(\omega_i)_{-\infty}^{-1}$ of sequences ω. We have a projection

$$\Pi : \Sigma_k^- \times \Sigma_k^+ \times I \to \Sigma_k^+ \times I.$$

Proposition 3.2.3. *Let μ^+ be an F^+-invariant probability measure with marginal ν^+. Then there exists an F-invariant probability measure μ with marginal ν and conditional measures*

$$\mu_\omega = \lim_{n\to\infty} f_{\sigma^{-n}\omega}^n \mu_{\sigma^{-n}\omega}^+, \tag{3.2.3}$$

ν-almost surely.

Let μ be an F-invariant probability measure with marginal ν. Then,

$$\mu^+ = \Pi\mu \tag{3.2.4}$$

is an F^+-invariant probability measure with marginal ν^+.

The correspondence $\mu \leftrightarrow \mu^+$ given by (3.2.3), (3.2.4) *is one-to-one. Furthermore, through these relations, F^+-invariant product measures $\mu^+ = \nu^+ \times m$ correspond one-to-one with F-invariant product measures $\mu = \nu^+ \times \vartheta$ on $\Sigma_k^+ \times (\Sigma_k^- \times I)$. The measure μ is ergodic if and only if μ^+ is ergodic.*

3.2.3 Lyapunov exponents

Our discussion of contractive dynamics in the next section requires the notion of Lyapunov exponents. The Lyapunov exponent of F^+ at a point $(\omega, x) \in \Sigma_k^+ \times I$ is

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| f'_{\omega_{n-1}}(f_\omega^{n-1}(x)) \cdots f'_{\omega_0}(x) \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'_{\sigma^i\omega}(f_\omega^i(x)) \right|,$$

in case the limit exists. Given an ergodic stationary measure m, by Birkhoff's ergodic theorem

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \ln \left| f'_{\omega_{n-1}}(f_\omega^{n-1}(x)) \cdots f'_{\omega_0}(x) \right| &= \int_{\Sigma_k^+} \int_0^1 \ln |f'_\omega(x)| \; dm(x) d\nu^+(\omega) \\ &= \int_0^1 \sum_{i=1}^k p_i \ln |f'_i(x)| \; dm(x),\end{aligned}$$

for $\nu^+ \times m$-almost all $(\omega, x) \in \Sigma_k^+ \times I$ (the value can be $-\infty$). The number on the right hand side is referred to as the Lyapunov exponent with respect to the ergodic stationary measure m. Since 0 is a fixed point of f_i for every i, the delta measure at 0 is a stationary measure. As $f'_i(0) = \rho_i$, we obtain for $x = 0$ that

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| f'_{\omega_{n-1}}(f_\omega^{n-1}(0)) \cdots f'_{\omega_0}(0) \right| = \sum_{i=1}^k p_i \ln(\rho_i),$$

for ν^+-almost all $\omega \in \Sigma_k^+$. We write

$$L(0) = \sum_{i=1}^k p_i \ln(\rho_i). \tag{3.2.5}$$

For $L(0) < 0$, the boundary point 0 is attracting on average [2]. The case $L(0) = 0$ is considered in [4]. We will be interested in the case $L(0) > 0$.

3.3 Synchronization

We formulate a general result on synchronization in iterated function systems of logistic maps. The result comes with assumptions on invariant measures and Lyapunov exponents; in addition to proving a general result we will provide checkable conditions for its assumptions. Figure 3.1 in the introduction provides an illustration of synchronized time series.

Theorem 3.3.1. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* $I = [0,1]$ *with* $0 < \rho_i < 4$. *Suppose*

(i) For some $1 \leqslant i_1 \leqslant k$, $\rho_{i_1} \in (1,3)$*: the map* f_{i_1} *possesses an attracting fixed point in* $(0,1)$ *with basin of attraction equal to* $(0,1)$*;*

(ii) $L(0) > 0$*: the fixed point at* $x = 0$ *is repelling on average;*

(iii) There is an ergodic stationary probability measure m *such that*

  (a) with respect to m*, the iterated function system has negative Lyapunov exponents;*

  (b) the iterated function system is minimal on supp $(m) \setminus \{0\}$.

For ν^+*-almost all* $\omega \in \Sigma_k^+$ *there is an open set* $W^s(\omega) \subset I$ *with* $m(W^s(\omega)) = 1$*, so that*

$$\lim_{n\to\infty} |f^n_\omega(x) - f^n_\omega(y)| = 0 \tag{3.3.1}$$

for $x, y \in W^s(\omega)$.

We prove Theorem 3.3.1 in Section 3.3.2, using Pesin theory for local stable sets in fibers $\{\omega\} \times I$ as developed in Section 3.3.1. Section 3.3.5 contains a more elementary proof of a somewhat stronger conclusion valid under stronger assumptions. In Sections 3.3.3 and 3.3.4 we present results on the existence of iterated function systems for which the conditions of Theorem 3.3.1, in particular the two parts of condition (iii), hold.

3.3.1 Local stable sets

Given an ergodic stationary measure with negative Lyapunov exponents, Pesin theory gives stable manifolds inside fibers $\{\omega\} \times I$. The following proposition extracts the statement for our setting. We provide a direct argument along the lines of [19, Lemma 2.2] or [7, Lemma A.1]. General statements for skew product systems with diffeomorphisms as fiber maps are in e.g. [9] or [23, Section 10]. Extensions to endomorphisms are treated in [26, Section 5].

Proposition 3.3.2. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 4$. *Suppose there is an ergodic stationary measure* m *so that the iterated function system has negative*

Lyapunov exponents with respect to m. For each $\zeta > 0$, there are $\eta > 0$, a set $\mathcal{A}^+ \subset \Sigma_k^+$ of positive measure $\nu^+(\mathcal{A}) > 1 - \zeta$ and constants $C > 0, 0 < \lambda < 1$, so that for $\omega \in \mathcal{A}^+$, there is an interval $B_\omega^s \subset I$ of length at least η with

$$|f_\omega^n(x) - f_\omega^n(y)| \leqslant C\lambda^n |x - y|$$

for $x, y \in B_\omega^s$.

Proof. By Birkhoff's ergodic theorem, there is a set of full $\nu^+ \times m$ measure so that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'_{\sigma^i\omega}(f_\omega^i(x))| < 0$$

for (ω, x) in this set. By Fubini's theorem there is a subset $B \subset I$ of full m-measure, so that for any $x \in B$ there is a set of full ν^+-measure $\Theta_x \subset \Sigma_k^+$ with

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'_{\sigma^i\omega}(f_\omega^i(x))| < 0, \ \forall \omega \in \Theta_x.$$

Take $x_0 \in B$. For $\varepsilon > 0$, $\omega \in \Theta_{x_0}$ and $i = 0, 1, \ldots$ write

$$a(\omega, i) = \ln \left(|f'_{\sigma^i\omega}(f_\omega^i(x_0))| + \varepsilon \right).$$

For every $\omega \in \Theta_{x_0}$ there is a small $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$

$$L_\varepsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} a(\omega, i) = \int_0^1 \sum_{i=1}^{k} p_i \ln \left(|f_i'(x)| + \varepsilon \right) \, dm(x) \tag{3.3.2}$$

and is negative (by monotone convergence). So, for ν^+-almost all $\omega \in \Sigma_k^+$ and $0 < \varepsilon \leqslant \varepsilon_0$, $\sum_{i=0}^{n-1} a(\omega, i)$ converges to $-\infty$ as $n \to \infty$ and

$$A(\omega) = \max\{0, \max_{n \geqslant 1} \sum_{i=0}^{n-1} a(\omega, i)\}$$

exists.

Take $0 < \varepsilon \leqslant \varepsilon_0$. There exists $\delta > 0$ so that if $|x - y| < \delta$, then

$$|f_j'(x)| \leqslant |f_j'(y)| + \varepsilon, \ \forall 1 \leqslant j \leqslant k.$$

Let B_ω^s be a neighborhood of x_0 with radius $\delta e^{-A(\omega)}$. For every x in B_ω^s, there exists $c_0 \in (x_0, x)$ such that

$$\big|f_\omega(x) - f_\omega(x_0)\big| = |f'_\omega(c_0)||x - x_0|.$$

Since $|c_0 - x_0| < \delta$ we find $|f'_\omega(c_0)| \leqslant |f'_\omega(x_0)| + \varepsilon$ and

$$|f_\omega(x) - f_\omega(x_0)| \leqslant e^{a(\omega,0)}|x - x_0|.$$

Note that $|f_\omega(x) - f_\omega(x_0)| < \delta e^{a(\omega,0)} e^{-A(\omega)} \leqslant \delta$. So with a similar reasoning as above,

$$|f_\omega^n(x) - f_\omega^n(x_0)| < e^{\sum_{i=0}^{n-1} a(\omega,i)} |x - x_0|,$$

for every $n \in \mathbb{N}$. Write $\Lambda_n = \frac{1}{n} \sum_{i=0}^{n-1} a(\omega, i)$. By (3.3.2), for every $\varepsilon' > 0$ there exists $N_{\varepsilon'} > 0$ (depending on ω and ε) such that for every $n > N_{\varepsilon'}$,

$$|\Lambda_n - L_\varepsilon| < \varepsilon'.$$

Since $L_\varepsilon < 0$ we can take $\varepsilon' > 0$ such that $L_\varepsilon + \varepsilon' < 0$. Now for $\omega \in \Theta_{x_0}$, $\lambda = e^{L_\varepsilon + \varepsilon'} < 1$ and $n > N_{\varepsilon'}$,

$$|f_\omega^n(x) - f_\omega^n(x_0)| < e^{n\Lambda_n} |x - x_0| < e^{n(L_\varepsilon + \varepsilon')} |x - x_0| < \lambda^n |x - x_0|.$$

Define

$$C'(\omega) = \max_{0 \leqslant n \leqslant N_{\varepsilon'}} \{ e^{\sum_{i=0}^{n-1} a(\omega,i)} \},$$

and

$$C(\omega) = \max\{1, \frac{C'(\omega)}{e^{N_{\varepsilon'}(L_\varepsilon + \varepsilon')}}\}.$$

Then for ν^+-almost every $\omega \in \Sigma_k^+$ and every $n \in \mathbb{N}$ we have

$$|f_\omega^n(x) - f_\omega^n(y)| \leqslant C(\omega) \lambda^n |x - x_0|.$$

The function $C(\omega)$ depends measurably on ω. So for every small $\zeta > 0$, by Lusin's theorem there is a compact set $\mathcal{A}^+ \subset \Sigma_k^+$ of measure $\nu^+(\mathcal{A}^+) > 1 - \zeta$ so that B_ω^s is of length at least $0 < \eta < \delta$ and the function C is continuous on $\mathcal{A}^+$. Therefore it is bounded by a constant C and the proposition is proved. □

By taking, in the proof of Proposition 3.3.2, $x_0 \in \operatorname{supp}(m)$, we find

$$\nu^+ \times m \left(\bigcup_{\omega \in \mathcal{A}^+} \{\omega\} \times B_\omega^s \right) > 0. \tag{3.3.3}$$

The following corollary is just a reformulation of Proposition 3.3.2 for the two sided skew product system $F : \Sigma^k \times I \to \Sigma^k \times I$, obtained by noting that the fiber coordinates of $F(\omega, x)$ do not depend on the past ω^- of $\omega = (\omega^-, \omega^+) \in \Sigma^- \times \Sigma^+$.

Corollary 3.3.3. *Take the assumptions of Proposition 3.3.2. For each $\zeta > 0$, there are $\eta > 0$, a set $\mathcal{A} = \Sigma_k^- \times \mathcal{A}^+ \subset \Sigma_k$ of positive measure $\nu(\mathcal{A}) > 1 - \zeta$ and constants $C > 0, 0 < \lambda < 1$, so that for $\omega \in \mathcal{A}$, there is an interval $B_\omega^s \subset I$ of length at least η with*

$$|f_\omega^n(x) - f_\omega^n(y)| \leqslant C\lambda^n \, |x - y| \tag{3.3.4}$$

for $x, y \in B_\omega^s$.

3.3.2 Proof of Theorem 3.3.1

Let us start with some comments on the proof strategy of Theorem 3.3.1. Proposition 3.3.2 gives local stable sets in fibers $\{\omega\} \times I$. Typical orbits enter these stable sets by ergodicity, resulting in local contraction properties. This will be combined with the existence of a large stable set for one of the logistic maps, given by item (i) in Theorem 3.3.1. We use a pullback argument so that we can apply the convergence (3.2.3) in Proposition 3.2.3. For this the extension to the two sided shift on Σ_k is needed. We refer to Section 3.3.5 for an argument that avoids this pullback argument, valid under stronger assumptions.

Proof of Theorem 3.3.1. We will deduce Theorem 3.3.1 from the existence of an invariant measurable graph Γ for the two sided system $F : \Sigma_k \times I \to \Sigma_k \times I$. The graph Γ is the graph of a measurable function $X : \Sigma_k \to I$ constructed in the following proposition. To prove its existence we follow the approach in [16].

Proposition 3.3.4. *Assume the conditions from Theorem 3.3.1. Let μ be the invariant measure for F corresponding to m as in Proposition 3.2.3. Then the conditional measure μ_ω on $\{\omega\} \times I$ is a delta measure for ν-almost all ω: there exists a measurable function $X : \Sigma_k \to I$ so that*

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega} m = \delta_{X(\omega)},$$

for ν-almost all ω, with convergence in the weak star topology.

Proof. Consider the map f_{i_1} from item (i) and its attracting fixed point at q_{i_1}. Note that

$$m(W^s(q_{i_1})) = m((0,1)) = 1.$$

Hence, for any $\varepsilon > 0$ there is a closed interval $R_\varepsilon \subset (0,1)$ with $m(R_\varepsilon) > 1-\varepsilon$. For an $\varepsilon > 0$ let $\Delta_\varepsilon \subset \mathcal{M}_I$ be the subset of probability measures on I that assign at least mass $1-\varepsilon$ to some point:

$$\Delta_\varepsilon = \{m \in \mathcal{M}_I;\, m(x) \geqslant 1-\varepsilon \,\text{for some}\, x \in I\}.$$

Note that Δ_ε's are closed subsets of $\mathcal{M}_I$.

Fix a small $\varepsilon > 0$ and take R_ε and Δ_ε as above. Consider $\mathcal{A} \subset \Sigma_k$, B^s_ω and $\eta > 0$ provided by Corollary 3.3.3.

Lemma 3.3.5. *There exists $L \in \mathbb{N}$ so that for each $\omega \in \mathcal{A}$, there exists $\mathcal{B}_{\omega^+} \subset \Sigma_k^-$ so that for $\gamma \in \mathcal{B}_{\omega^+} \times \{\omega^+\}$, $f^L_{\sigma^{-L}\gamma}$ maps R_ε into B^s_ω.*

Proof. For any small $r > 0$, there is a sufficiently large L_1 iterate of f_{i_1} that maps R_ε into a neighborhood $(q_{i_1} - r, q_{i_1} + r)$ of q_{i_1}. Since IFS ($\mathbb{F}$) is minimal on $\operatorname{supp}(m)\setminus\{0\}$, recall item (iii), every open set in $\operatorname{supp}(m)$ has intersection with the set $\bigcup_{\omega\in\Sigma_k} f^n_\omega(q_{i_1})$ for some $n \geqslant 0$. Hence for $\eta > 0$ there is an integer L_2 so that for any open interval B of diameter η with positive measure, there are symbols $j_{L_2}, \dots, j_1$ so that $f_{j_{L_2}} \circ \cdots \circ f_{j_1}(q_{i_1}) \in B$. Combining the above statements, there is a composition $f_{j_{L_2}} \circ \cdots \circ f_{j_1} \circ f_{i_1}^{L_1}$ that maps R_ε into B^s_ω. We let $\mathcal{B}_{\omega^+}$ consist of the sequences in Σ_k^- that end with symbols ${i_1}^{L_1} j_1 \dots j_{L_2}$. We have that L_1, L_2 are uniformly bounded but need not be constant in ω. We can get $L_1 + L_2$ to be constant on $\mathcal{A}$ by adjusting the number of iterates L_1. This proves the lemma with $L = L_1 + L_2$. □

Observe that $\nu_-(\mathcal{B}_{\omega^+})$ is uniformly bounded away from zero. Consequently the union

$$\mathcal{B} = \cup_{\omega \in \mathcal{A}} \mathcal{B}_{\omega^+} \times \{\omega^+\} \tag{3.3.5}$$

has positive measure: $\nu(\mathcal{B}) > 0$. By ergodicity of ν, for ν-almost all ω, its orbit under σ^{-1} intersects $\mathcal{B}$. So for such ω and every small $\varepsilon > 0$, Lemma 3.3.5 and Corollary 3.3.3 yield

$$\liminf_{n\to\infty} d_{\mathcal{M}_I}\left(f^n_{\sigma^{-n}\omega} m, \Delta_\varepsilon\right) = 0 \tag{3.3.6}$$

(recall from Section 3.2.1 that $d_{\mathcal{M}_I}$ is a metric on $\mathcal{M}_I$ generating the weak star topology). Letting $\varepsilon \to 0$, we observe that Δ_ε converges to the set of δ-measures in $\mathcal{M}_I$. Therefore, by (3.3.6) and Proposition 3.2.3,

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega} m = \delta_{X(\omega)}$$

for a measurable function $X : \Sigma_k \to I$. □

The synchronization property (3.3.1) will be obtained as a consequence of the existence of the invariant graph Γ and the negative sign of the Lyapunov exponents. For ν-almost all $\omega \in \Sigma_k$, the Lyapunov exponents at $(\omega, X(\omega))$ exist and are strictly negative. Write $W^s(X(\omega))$ for the stable set of $X(\omega)$ inside the fiber $\{\omega\} \times I$;

$$W^s(X(\omega)) = \{y \in I \; ; \; \lim_{n\to\infty} |f^n_\omega(y) - X(\sigma^n\omega)| = 0\}.$$

The theory of nonuniform hyperbolicity, as in Proposition 3.3.2 and Corollary 3.3.3, yields the following. Write $D_\delta(X(\omega))$ for the δ-ball around $X(\omega)$. Then for all $\varepsilon > 0$ there is $\delta > 0$ so that

$$S(\delta) = \{\omega \in \Sigma_k \; ; \; D_\delta(X(\omega)) \subset W^s(X(\omega))\}$$

satisfies

$$\nu(S(\delta)) > 1 - \varepsilon. \tag{3.3.7}$$

Once orbits are in a δ-ball $D_\delta(X(\omega))$ for $\omega \in S(\delta)$, distances to the orbit of $X(\omega)$ decrease to zero, which we may assume to happen at a uniform rate as in (3.3.4).

Proposition 3.3.6. *For ν-almost all $\omega \in \Sigma_k$, we have that $W^s(X(\omega))$ is open with $m(W^s(X(\omega))) = 1$.*

Proof. We follow [16]. For ν-almost all $\omega \in \Sigma_k$, $W^s(X(\omega))$ is open. Indeed, take $y \in W^s(X(\omega))$. For ν-almost all $\omega \in \Sigma_k$, $\sigma^n\omega \in S(\delta)$ for infinitely many positive integers n. We may take n large so that $\sigma^n\omega \in S(\delta)$ and $f^n_\omega(y) \in D_\delta(X(\sigma^n\omega)) \subset W^s(X(\sigma^n\omega))$. By continuity of $f_1, \dots, f_k$, a small neighborhood of y lies in $W^s(X(\omega))$.

We have that $f^n_{\sigma^{-n}\omega} m$ converges to $\delta_{X(\omega)}$, ν-almost surely. This implies convergence in measure, and since σ leaves ν invariant, also that $f^n_\omega m$ converges to $\delta_{X(\sigma^n\omega)}$ in measure. That is, for any $\varepsilon > 0$,

$$\nu\{\omega \in \Sigma_k \; ; \; d_{\mathcal{M}_I}(f^n_\omega m, \delta_{X(\sigma^n\omega)}) > \varepsilon\} \to 0$$

as $n \to \infty$. This in turn implies that for some subsequence $n_k \to \infty$,

$$\nu\{\omega \in \Sigma_k \; ; \; \lim_{k\to\infty} d_{\mathcal{M}_I}(f^{n_k}_\omega m, \delta_{X(\sigma^{n_k}\omega)}) = 0\} = 1 \tag{3.3.8}$$

(see e.g. [27, Theorem II.10.5]). We combine this with the existence of stable sets around $X(\sigma^n\omega)$ to prove that $d_{\mathcal{M}_I}(f^n_\omega m, \delta_{X(\sigma^n\omega)}) \to 0$ almost surely. In more detail, let

$$\Gamma(\hat{\delta}, N) = \{\omega \in \Sigma_k \; ; \; d_{\mathcal{M}_I}(f^N_\omega m, \delta_{X(\sigma^N\omega)}) < \hat{\delta}\}.$$

Now (3.3.8) implies that for any given $\varepsilon > 0$, $\hat{\delta} > 0$, there is $N > 0$ with

$$\nu(\Gamma(\hat{\delta}, N)) > 1 - \varepsilon. \tag{3.3.9}$$

A measure is close to a delta measure if most of the measure is in a small ball: for any ε, δ there is $\hat{\delta} > 0$ so that $d_{\mathcal{M}_I}(\mu, \delta_x) < \hat{\delta}$ implies $\mu(D_\delta(x)) > 1 - \varepsilon$. So (3.3.9) gives that for any $\varepsilon > 0, \delta > 0$ there exists $N > 0$ so that

$$\nu\{\omega \in \Sigma_k \; ; \; f^N_\omega m(D_\delta(X(\sigma^N\omega))) > 1 - \varepsilon\} > 1 - \varepsilon.$$

Combining this with (3.3.7) we get that for all $\varepsilon > 0$, $\delta > 0$ there exists $N > 0$ so that the set

$$T_\varepsilon = \{\omega \in \Sigma_k \; ; \; \text{for all } n \geqslant N, f^n_\omega m(D_\delta(X(\sigma^n\omega))) > 1 - \varepsilon\}$$

satisfies $\nu(T_\varepsilon) > 1 - \varepsilon$. We thus find that for all $\delta > 0$,

$$\nu\left(\{\omega \in \Sigma_k \; ; \; \lim_{n\to\infty} f^n_\omega m(D_\delta(X(\sigma^n\omega))) = 1\}\right) = 1.$$

Consequently, for ν-almost all $\omega \in \Sigma_k$,

$$\lim_{n\to\infty} d_{\mathcal{M}_I}(f^n_\omega m, \delta_{X(\sigma^n\omega)}) = 0$$

and

$$m(W^s(X(\omega))) = 1.$$

$\square$

The synchronization property (3.3.1) holds for $x, y \in W^s(X(\nu))$ with $\pi\nu = \omega$. $\square$

Observe that as a consequence of Proposition 3.3.6,

$$\operatorname{supp}(m) \subset \overline{W^s(X(\omega))}$$

for ν-almost all $\omega \in \Sigma_k$.

3.3.3 Stationary measures with negative Lyapunov exponents

In this section we investigate stationary measures with full measure in $(0,1)$ and in particular stationary measures for which the iterated function system has negative Lyapunov exponents. Some specific cases of the existence of stationary measures, in particular when a reduction to iterated function systems of monotone interval maps is possible, are contained in [6], see also [5, Section 4.4], and further [8].The existence of a stationary measure with $m(\{0\}) = 0$ under our assumptions is proved in [2, Theorem 2]; following [12, 13] we provide an alternative argument and a bound on the measure near the boundary point 0.

Proposition 3.3.7. *Consider an iterated function system IFS $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, of logistic maps $f_i(x) = \rho_i x(1-x)$ on I with $0 < \rho_i < 4$. Assume $L(0) > 0$. Then there exists an ergodic stationary measure m with $m(\{0\}) = 0$. Moreover, for some $c > 0$ and $\alpha > 0$, $m\big([0,x)\big) \leqslant cx^\alpha$.*

Proof. Recall from Section 3.2.1 that $\mathcal{T}$ defines a map from the space of probability measures $\mathcal{M}_I$ to itself and that a measure m on I is a stationary measure for the iterated function system precisely if $\mathcal{T}m = m$. The construction of stationary measures here is similar to the construction of stationary measures in [13, Lemma 3.2] and [12, Proposition 4.1] for interval diffeomorphisms.

Note that $\rho_i \in (0,4)$ implies that $\max_{1\leqslant i\leqslant k} \rho_i < 4$. Since $f_i \leqslant \rho_i/4$ on I we find that any stationary measure has support contained in $[0, \max_{1\leqslant i\leqslant k} \rho_i/4] \subset [0,1)$. For small $0 < \alpha < 1$ and $q > 0$ define

$$\mathcal{N}_c = \{m \in \mathcal{M}_I;\ \operatorname{supp}(m) \subset [0, \max_{1\leqslant i\leqslant k} \rho_i/4] \text{ and } \forall\ 0 \leqslant x \leqslant q,\ m\big([0,x)\big) \leqslant cx^\alpha\}. \tag{3.3.10}$$

This defines a closed subset of $\mathcal{M}_I$. The condition on the measure of small intervals $[0,x)$ excludes stationary measures that assign positive measure to $\{0\}$. Note that $\mathcal{N}_c$ depends on α and q; but this dependence is not included in the notation. Similar to the proof in [13] and [12] one can show that there exist positive α and q close to 0 such that $\mathcal{T}(\mathcal{N}_c) \subset \mathcal{N}_c$. By the Krylov-Bogolyubov averaging method, for a measure $m \in \mathcal{N}_c$ there is a subsequence of $\{\frac{1}{n}\sum_{r=0}^{n-1} \mathcal{T}^r m\}_{n\in\mathbb{N}}$ that is convergent to a probability measure $\hat{m} \in \mathcal{N}_c$ such that $\mathcal{T}\hat{m} = \hat{m}$. It is proved in [13] and [12] that also an ergodic stationary measure in $\mathcal{N}_c$ exists. $\square$

Proposition 3.3.8. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I. *Assume* $L(0) > 0$ *and for all* $1 \leqslant i \leqslant k$, $\rho_i \in (0, 8/3)$. *Then there is a stationary measure* m *with* $m(\{0\}) = 0$ *and so that IFS* $(\mathbb{F})$ *has negative Lyapunov exponents with respect to* m.

Proof. We will demonstrate that with respect to the ergodic stationary measure m from Proposition 3.3.7, IFS $(\mathbb{F})$ has negative Lyapunov exponents. We first derive the identity

$$L(0) = -\int_0^1 \ln(1-x)\, dm(x), \tag{3.3.11}$$

also contained in [2, Theorem 1]. For an integrable function φ on I,

$$\int_0^1 \sum_{i=1}^k p_i \varphi \circ f_i \, dm = \int_0^1 \varphi \, dm. \tag{3.3.12}$$

One obtains from Proposition 3.3.7 that the function $x \mapsto \ln(x)$ is integrable. For completeness we provide a proof of this elementary fact.

Lemma 3.3.9. *The function* $x \mapsto \ln(x)$ *is integrable for the measure* m.

Proof. Let $h(x) = cx^\alpha$, thus $h^{-1}(x) = \frac{1}{c^{1/\alpha}} x^{1/\alpha}$, and consider $\tilde{m} = hm$. Observe that

$$\tilde{m}([0, x)) \leqslant x. \tag{3.3.13}$$

Then $\int_J \ln(x)\, dm(x) = \int_{h^{-1}(J)} \ln(h^{-1}(x))\, d\tilde{m}(x)$ for suitable Borel sets J. It suffices to check that $\ln(h^{-1}(x))$ is integrable for the measure $\tilde{m}$. By integrability of $\ln(h^{-1}(x))$ for Lebesgue measure, we have that for any $\varepsilon > 0$ one can find $0 = a_{N+1} < a_N < \ldots < a_1$, so that

$$\sum_{i=1}^N |\ln(h^{-1}(a_i))| |a_i - a_{i+1}| \leqslant \int_0^{a_1} |\ln(h^{-1}(x))|\, dx + \varepsilon.$$

Denote $I_i = [a_{i+1}, a_i)$ and $|I_i| = |a_i - a_{i+1}|$. If $\tilde{m}(I_1) - |I_1| \leqslant 0$, then

$$|\ln(h^{-1}(a_1))| \tilde{m}(I_1) \leqslant |\ln(h^{-1}(a_1))| |I_1|.$$

If $\tilde{m}(I_1) - |I_1| > 0$, then

$$\begin{aligned} |\ln(h^{-1}(a_1))| \tilde{m}(I_1) &\leqslant |\ln(h^{-1}(a_1))| |I_1| + |\ln(h^{-1}(a_1))| (\tilde{m}(I_1) - |I_1|) \\ &\leqslant |\ln(h^{-1}(a_1))| |I_1| + |\ln(h^{-1}(a_2))| (\tilde{m}(I_1) - |I_1|). \end{aligned}$$

If also $\tilde{m}(I_2) - |I_2| + \tilde{m}(I_1) - |I_1| > 0$, then

$$\sum_{i=1}^2 |\ln(h^{-1}(a_i))| \tilde{m}(I_i) \leqslant \sum_{i=1}^2 |\ln(h^{-1}(a_i))| |I_i| + |\ln(h^{-1}(a_2))| \sum_{i=1}^2 (\tilde{m}(I_i) - |I_i|).$$

Continuing this reasoning, employing that $\sum_{i=1}^{N} \tilde{m}(I_i) - |I_1| \leqslant 0$ holds by (3.3.13), shows

$$\begin{aligned}\sum_{i=1}^{N} |\ln(h^{-1}(a_i))| \tilde{m}(I_i) &\leqslant \sum_{i=1}^{N} |\ln(h^{-1}(a_i))||I_i| \\ &\leqslant \int_0^{a_1} |\ln(h^{-1}(x))|\, dx + \varepsilon.\end{aligned}$$

This estimate proves integrability of $\ln(h^{-1}(x))$ for the measure $\tilde{m}$. □

Applying identity (3.3.12) to $\varphi(x) = \ln(x)$,

$$\begin{aligned}\int_0^1 \sum_{i=1}^{k} p_i \ln(\rho_i)\, dm(x) + \int_0^1 \sum_{i=1}^{k} p_i \ln(x)\, dm(x) \quad &+ \int_0^1 \sum_{i=1}^{k} p_i \ln(1-x)\, dm(x) \\ = \int_0^1 \ln(x)\, dm(x).&\end{aligned}$$

This computation proves (3.3.11).

On the other hand, the Lyapunov exponent with respect to m is

$$\begin{aligned}\int_0^1 \sum_{i=1}^{k} p_i \ln|f_i'(x)|\, dm(x) &= \int_0^1 \sum_{i=1}^{k} p_i \ln|\rho_i(1-2x)|\, dm(x) \\ &= \int_0^1 \sum_{i=1}^{k} p_i \ln(\rho_i)\, dm(x) + \int_0^1 \sum_{i=1}^{k} p_i \ln|1-2x|\, dm(x) \\ &= \int_0^1 L(0)\, dm(x) + \int_0^1 \ln|1-2x|\, dm(x) \\ &= L(0) + \int_0^1 \ln|1-2x|\, dm(x). \end{aligned} \tag{3.3.14}$$

Combining (3.3.11) and (3.3.14), the condition

$$\int_0^1 \sum_{i=1}^{k} p_i \ln|f_i'(x)|\, dm(x) < 0$$

yields

$$\int_0^1 \ln|1-2x|\, dm(x) - \int_0^1 \ln(1-x)\, dm(x) = \int_0^1 \ln\left(\frac{|1-2x|}{1-x}\right) dm(x) < 0.$$

A sufficient condition for this inequality to hold is $\ln\left(\frac{|1-2x|}{1-x}\right) < 0$ on $\operatorname{supp}(m)$; i.e. $|1-2x| < 1-x$ for every $x \in \operatorname{supp}(m)$. This means that $\operatorname{supp}(m) \subset [0, \frac{2}{3})$ (note that we have $m(\{0\}) = 0$). Since f_i assumes its maximum at $f_i(1/2) = \rho_i/4$, negative Lyapunov exponents occur when $\rho_i < \frac{8}{3}$ for all $1 \leqslant i \leqslant k$. or all $1 \leqslant i \leqslant k$, in which case $\operatorname{supp}(m) \subset [0, 3/4]$. □

Akin to Singer's theorem [22, Chapter II, Theorem 6.1], stating that a logistic map can have at most one periodic attractor, we find that an iterated function system of logistic maps can have at most one stationary measure with negative Lyapunov exponents.

Proposition 3.3.10. *Consider an iterated function system IFS*$(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 4$. *The iterated function system IFS*$(\mathbb{F})$ *possesses at most one stationary measure with negative Lyapunov exponents.*

Proof. Suppose m is an ergodic stationary measure with negative Lyapunov exponents, so that $\nu^+ \times m$ is an invariant measure for F^+. Consider the extension F, and an ergodic invariant measure μ, with marginal ν on Σ_k, corresponding to m as provided by Proposition 3.2.3. Let $\mathcal{A} \subset \Sigma_k$ be the set of positive measure from Corollary 3.3.3.Write $\mathbb{X} \subset \Sigma_k \times I$ for the collection

$$\mathbb{X} = \bigcup_{\omega \in \mathcal{A}} \{\omega\} \times B^s_\omega$$

of stable sets. Similarly to (3.3.3) we have $\mu(\mathbb{X}) > 0$. For μ-almost all (ω, x), (3.2.2) holds, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i_\omega(x)} = m.$$

Further, by ergodicity, for μ-almost all (ω, x), the orbit under F intersects $\mathbb{X}$ infinitely often. Consider two iterates $F^p(\omega, x)$ and $F^n(\omega, x)$, $n > p$, that lie in $\mathbb{X}$.

Now for $n-p$ large enough, $f^{n-p}_{\sigma^p\omega}$ maps $B^s_{\sigma^p\omega}$ into $B^s_{\sigma^n\omega}$ and has small diameter since

$$\left| f^{n-p}_{\sigma^p\omega}(B^s_{\sigma^p\omega}) \right| \leqslant C\lambda^{n-p} \tag{3.3.15}$$

for some $C > 0, 0 < \lambda < 1$.

Each composition f^i_ω has negative Schwarzian derivative. By the minimum principle, see [22, Chapter II, Lemma 6.1], if $f^{n-p}_{\sigma^p\omega}$ does not have a critical point on $B^s_{\sigma^p\omega}$, then the minimum of $|\left(f^{n-p}_{\sigma^p\omega}\right)'|$ on $B^s_{\sigma^p\omega}$ is assumed at a boundary point. We may write $B^s_{\sigma^p\omega} = (b_l, b_r)$, and assume then that b_r is this boundary point. Let $\hat{B}_{\sigma^p\omega} = (b_l, c_r)$ be the interval, extended maximally on the side of b_r, so that $f^{n-p}_{\sigma^p\omega}$ is monotone on $\hat{B}_{\sigma^p\omega}$. So $f^i_{\sigma^p\omega} c_r = 1/2$ for some $0 \leqslant i < n-p$. Because

$$\left| \left(f^{n-p}_{\sigma^p\omega}\right)'\Big|_{\hat{B}_{\sigma^p\omega}} \right| \leqslant \left| \left(f^{n-p}_{\sigma^p\omega}\right)'\Big|_{B^s_{\sigma^p\omega}} \right|$$

and $|B^s_{\sigma^p\omega}| \geqslant \eta$, we find that there is $\tilde{C} \geqslant 1$ so that

$$\left| f^{n-p}_{\sigma^p\omega}(\hat{B}_{\sigma^p\omega}) \right| \leqslant \tilde{C} \left| f^{n-p}_{\sigma^p\omega}(B^s_{\sigma^p\omega}) \right|.$$

Because of this and (3.3.15), we find that for $n-p$ large enough,

$$f^{n-p}_{\sigma^p\omega}(\hat{B}_{\sigma^p\omega}) \subset B^s_{\sigma^n\omega}.$$

We conclude that always there is $r \geqslant 0$ so that

$$\lim_{i\to\infty} \left| f^i_{\sigma^r\omega}(f^r_\omega(x)) - f^i_\omega(1/2)\right| = 0,$$

and so

$$\lim_{i\to\infty} \left| f^i_\tau(y)) - f^i_\tau(1/2)\right| = 0,$$

for $(\tau, y) = F^r(\omega, x)$. Observe that (3.2.2) holds when replacing (ω, x) by (τ, y):

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i_\tau}(y) = m.$$

Take $\varepsilon > 0$ and let $\psi : I \to \mathbb{R}$ be a continuous function. By uniform continuity, there is $\delta > 0$ so that $|\psi(y_1) - \psi(y_2)| < \varepsilon$ whenever $|y_1 - y_2| < \delta$. Let $N > 0$ be large enough so that $|f^i_\tau(1/2) - f^i_\tau(y)| < \delta$ for $i \geqslant N$, and let then n be large enough so that $\frac{1}{n} 2N \max_{z\in[0,1]} |\psi(z)| < \varepsilon$. Then

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=0}^{n-1} \left(\psi(f^i_\tau(1/2)) - \psi(f^i_\tau(y))\right)\right| &\leqslant \frac{1}{n}\sum_{i=0}^{N-1} \left|\left(\psi(f^i_\tau(1/2)) - \psi(f^i_\tau(y))\right)\right| \\
&\quad + \lim_{n\to\infty} \frac{1}{n}\sum_{i=N}^{n-1} \left|\left(\psi(f^i_\tau(1/2)) - \psi(f^i_\tau(y))\right)\right| \\
&\leqslant \frac{2N \max_{z\in[0,1]} |\psi(z)|}{n} + \varepsilon \\
&< 2\varepsilon.
\end{aligned}$$

As ε is arbitrary, this shows

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \psi(f^i_\tau(1/2)) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \psi(f^i_\tau(y)),$$

for each continuous function $\psi : I \to \mathbb{R}$, i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i_\tau}(1/2) = m.$$

This proves the proposition. □

The argument in the above proof makes clear that for ν^+-almost all $\omega \in \Sigma^+_k$, $(\omega, 1/2)$ is a generic point for an ergodic invariant measure $\nu^+ \times m$ (i.e. (3.2.2) holds for $x = 1/2$) when one assumes negative Lyapunov exponents.

Write

$$\Delta^\ell = \left\{ (x_1, \ldots, x_\ell) \in [0,1]^\ell \ ; \ \sum_{i=1}^{\ell} x_i = 1 \right\}$$

for the standard ℓ-simplex.

Proposition 3.3.11. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 4$. *Suppose IFS* $(\mathbb{F})$ *admits a unique stationary measure* m *with* $m(\{0\}) = 0$. *Assume that* $L(0) > 0$ *and that IFS* $(\mathbb{F})$ *has negative Lyapunov exponents with respect to* m. *Let* $\ell \geqslant k$ *and* $\rho_{k+1}, \ldots, \rho_\ell \in (0,4)$. *There are neighborhoods* V *of* $(\rho_1, \ldots, \rho_\ell)$ *in* $(0,4)^\ell$ *and* U *of* $(p_1, \ldots, p_k, 0, \ldots, 0)$ *in* Δ^ℓ, *so that for elements of* U, V, *the corresponding iterated function system has a unique stationary measure that assigns measure* 0 *to* $\{0\}$ *and has negative Lyapunov exponents.*

Proof. Take a sequence $f_1^i, \ldots, f_\ell^i$ of logistic maps converging to $f_1, \ldots, f_\ell$ as $i \to \infty$. That is, with $f_j^i(x) = \rho_j^i x(1-x)$ and $f_j(x) = \rho_j x(1-x)$, we assume $\rho_j^i \to \rho_i$ as $i \to \infty$. Let also the probabilities p_j^i with which f_j^i is chosen, converge to p_j.

By Proposition 3.3.7 there is a stationary measure m_i for IFS $\left(\{f_1^i, \ldots, f_\ell^i\}\right)$ with $m_i(\{0\}) = 0$. By Lemma 3.2.1, any limit point of m_i in $\mathcal{M}_I$ is a stationary measure for IFS $(\mathbb{F})$. Further, there is a fixed space $\mathcal{N}_c$ as in (3.3.10) so that $m_i \in \mathcal{N}_c$ for all i large enough. Therefore m_i can not converge to a measure that assigns positive measure to $\{0\}$. We conclude that $m_i \to m$ as $i \to \infty$. The Lyapunov exponents with respect to m_i are therefore negative for large i. Proposition 3.3.10 implies that for i large, m_i is the unique stationary measure for IFS $(\{f_1^i, \ldots, f_\ell^i\})$ that assigns measure 0 to $\{0\}$. □

The following corollary, together with the statements on minimality in Section 3.3.4, allows the construction of iterated function systems for which Theorem 3.3.1 holds and that include any given logistic map $f_i(x) = \rho_i x(1-x)$ with $0 < \rho_i < 4$.

Corollary 3.3.12. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 8/3$. *Suppose* $L(0) > 0$. *Let* $\ell \geqslant k$ *and* $\rho_{k+1}, \ldots, \rho_\ell \in (0,4)$. *There are neighborhoods* V *of* $(\rho_1, \ldots, \rho_\ell)$ *in* $(0,4)^\ell$ *and* U *of* $(p_1, \ldots, p_k, 0, \ldots, 0)$ *in* Δ^ℓ, *so that for elements of* U, V, *the corresponding iterated function system has a unique stationary measure that assigns measure* 0 *to* $\{0\}$ *and has negative Lyapunov exponents.*

Proof. By Proposition 3.3.8, IFS$(\{f_1, \ldots, f_k\})$ admits a stationary measure m with $m(\{0\}) = 0$ and with negative Lyapunov exponents. Take a sequence $f_1^i, \ldots, f_\ell^i$ of logistic maps as in the proof of Proposition 3.3.11. As in the proof of Proposition 3.3.11 there are stationary measures m_i of IFS $(\{f_1^i, \ldots, f_\ell^i\})$ and there is a fixed space $\mathcal{N}_c$ as in (3.3.10) so that $m \in \mathcal{N}_c$ and $m_i \in \mathcal{N}_c$ for all i large enough. By Proposition 3.3.10, m is the unique stationary measure for IFS $(\mathbb{F})$ with negative Lyapunov exponents. The proof of Proposition 3.3.8 shows that any stationary measure for IFS $(\mathbb{F})$ in $\mathcal{N}_c$ has negative Lyapunov exponents, so that m is the unique stationary measure in $\mathcal{N}_c$. By Lemma 3.2.1, $m_i \to m$ as $i \to \infty$. □

3.3.4 Minimal iterated function systems

Sufficient conditions for minimality of the iterated function system on supp (m) are in the following result.

Proposition 3.3.13. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \dots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* I *with* $0 < \rho_i < 4$. *Assume* $L(0) > 0$. *Under any of the following conditions, the iterated function system IFS* $(\mathbb{F})$ *admits an ergodic stationary measure* m *with* $m(\{0\}) = 0$ *so that it acts minimally on supp* $(m) \setminus \{0\}$.

(a) There exists j, $1 \leqslant j \leqslant k$, *with* $\rho_j \in (0,1)$;

(b) There exists j_1, j_2, $1 \leqslant j_1, j_2 \leqslant k$, *with* $6/5 < \rho_{j_1} < \rho_{j_2} < 3/2$.

Under condition (a),

$$\mathit{supp}\,(m) = [0, M],$$

where $M = \max_{1 \leqslant i \leqslant k} f_i(1/2) = \max_{1 \leqslant i \leqslant k} \frac{\rho_i}{4}$.

Proof. Take a stationary measure m as provided by Proposition 3.3.7. Since $[0, M]$ is invariant for IFS $(\mathbb{F})$ and $f_i(x) \in [0, M]$ for every $x \in I$, $1 \leqslant i \leqslant k$, we have supp $(m) \subset [0, M]$.

We begin the proof with item (a) and will prove that the iterated function system is minimal on $(0, M]$ and that supp $(m) = [0, M]$. Consider f_j with $\rho_j \in (0,1)$. Since it is attracting at $x = 0$ the orbits can get arbitrarily close to 0. Hence, for every small $\delta > 0$ we have $m\big((0,\delta)\big) > 0$ and $\min \operatorname{supp}(m) = 0$. Applying the following lemma, which relates to Il′yashenko's work [18], we will find that the iterated function system is minimal on $(0, M]$.

Lemma 3.3.14. *For diffeomorphisms* $f, g : I \to I$ *fixing the boundary point* 0, *assume that* $\lambda = f'(0) < 1$, $\mu = g'(0) > 1$, *and*

$$\frac{f''(0)}{\lambda^2 - \lambda} \neq \frac{g''(0)}{\mu^2 - \mu}. \tag{3.3.16}$$

Then there is an interval $(0, u] \subset (0, 1)$ *such that all orbits of the iterated function system generated by* f, g *restricted to* $(0, u]$ *are dense in it.*

Proof. See [12, Proposition 2.1]. □

Recall the logistic map f_{i_1} with $\rho_{i_1} \in (1,3)$ from item (i) in Theorem 3.1. We know that $f_j'(0) < 1$, $f_{i_1}'(0) > 1$. The inequality (3.3.16) for $f = f_{i_1}$ and $g = f_j$ reads

$$\frac{-2\rho_j}{\rho_j^2 - \rho_j} \neq \frac{-2\rho_{i_1}}{\rho_{i_1}^2 - \rho_{i_1}}$$

and is satisfied. Hence Lemma 3.3.14 holds for $f = f_j$ and $g = f_{i_1}$ and some $(0, u] \subset (0, \frac{1}{2}]$.

In order to prove minimality of IFS $(\mathbb{F})$ on $(0, M]$ we need to show that for every $x \in (0, M]$ and every open interval $J' \subset (0, M]$ there is a composition h_1 of the f_i's, $1 \leqslant i \leqslant k$, such that $h_1(x) \in J'$. Let us assume that ρ_t is the maximal parameter value, so that $M = \rho_t/4$. Note that there is an interval $K \subset (0, u]$ such that a finite number of iterations of K by f_t covers $(0, M]$. Consider a small interval

$J \subset K$ that is mapped inside J' by an iterate of f_t, say $f_t^{n_1}(J) \subset J'$. Since 0 is attracting for f_j, there is $n_2 \in \mathbb{N}$ such that $f_j^{n_2}(x) \in (0, u]$. By Lemma 3.3.14 there is a map $h \in \mathrm{IFS}\,(\{f_j, f_{i_1}\})$ so that $h(f_j^{n_2}(x)) \in J$. Hence, $f_t^{n_1}(h(f_j^{n_2}(x))) \in J'$ and we can take $h_1 = f_t^{n_1} \circ h \circ f_j^{n_2}$. We conclude that IFS $(\mathbb{F})$ is minimal on $(0, M]$. This implies that supp $(m) = [0, M]$, compare [20, Proposition 5].

Now we establish the statement for item (b). Recall that f_i has a fixed point at $q_i = \frac{\rho_i - 1}{\rho_i}$. Note that the interval $R = [q_{j_1}, q_{j_2}]$ is invariant, since q_{j_1} and q_{j_2} are only attracting fixed points for f_{j_1} and f_{j_2} respectively. Also, the maps f_{j_1} and f_{j_2} are strictly increasing on R. We claim that

(I) $f_{j_1}(R) \cup f_{j_2}(R) = R$,

(II) f_{j_1} and f_{j_2} are contractions on R.

If

$$f_{j_1}(q_{j_2}) > f_{j_2}(q_{j_1}),$$

then $f_{j_1}(R) \cup f_{j_2}(R) = R$. Working out gives

$$\rho_{j_1} \frac{1}{\rho_{j_2}} \frac{\rho_{j_2} - 1}{\rho_{j_2}} > \rho_{j_2} \frac{1}{\rho_{j_1}} \frac{\rho_{j_1} - 1}{\rho_{j_1}},$$

that is

$$\frac{\rho_{j_1}^3}{\rho_{j_1} - 1} > \frac{\rho_{j_2}^3}{\rho_{j_2} - 1}.$$

This inequality holds for $1 < \rho_{j_1} < \rho_{j_2} < 3/2$.

Note that $0 < f'_{j_1} < 1$ on R. The minimum of f'_{j_2} on R is assumed at q_{j_1} and equals $\rho_{j_2}(\frac{2}{\rho_{j_1}} - 1)$. Assuming this to be smaller than 1, assures that f_{j_1} and f_{j_2} are contractions on R. This holds for $6/5 < \rho_{j_1} < \rho_{j_2} < 3/2$. It follows that IFS $(\{f_{j_1}, f_{j_2}\})$ acts minimally on the interval R. One necessarily has $R \subset$ supp (m). From this we can conclude that IFS $(\mathbb{F})$ acts minimally on supp $(m) \setminus \{0\}$. □

3.3.5 Forward convergence

We provide a simpler proof of synchronization, valid for specific cases, without using extensions to two sided time and pullback convergence techniques.

Theorem 3.3.15. *Consider an iterated function system IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, *of logistic maps* $f_i(x) = \rho_i x(1-x)$ *on* $I = [0,1]$ *with* $1 < \rho_i < 4$. *Suppose*

(i) For some $1 \leqslant i_1 \leqslant k$, $\rho_{i_1} \in (1,3)$*: the map* f_{i_1} *possesses an attracting fixed point in* $(0,1)$ *with basin of attraction equal to* $(0,1)$*;*

(ii) There is an ergodic stationary probability measure m *such that*

(a) with respect to m*, the iterated function system has negative Lyapunov exponents;*

(b) the iterated function system is minimal on supp(m).

For ν^+-almost all $\omega \in \Sigma_k^+$,

$$\lim_{n\to\infty} |f_\omega^n(x) - f_\omega^n(y)| = 0 \tag{3.3.17}$$

for $x, y \in (0,1)$.

Proof. It is easily seen that the interval $J = [j_l, j_r]$ with

$$\begin{aligned} j_r &= \max_{1\leqslant i\leqslant k} f_i(1/2), \\ j_l &= \min_{1\leqslant j\leqslant k} f_j(j_r), \end{aligned}$$

is forward invariant for IFS $(\mathcal{F})$. Every $x \in (0,1)$ is eventually mapped into J. It follows that

$$\operatorname{supp}(m) \subset J.$$

So the support $\operatorname{supp}(m)$ of the stationary measure m is disjoint from 0 and 1.

Lemma 3.3.16. *There are a set $\mathcal{C}^+ \subset \Sigma_k$ of positive measure $\nu^+(\mathcal{C}) > 0$ and constants $C > 0, 0 < \lambda < 1$, so that for $\omega \in \mathcal{C}^+$, there is an interval $B_\omega^s \subset I$ with $J \subset B_\omega^s$ and*

$$|f_\omega^n(x) - f_\omega^n(y)| \leqslant C\lambda^n\, |x-y| \tag{3.3.18}$$

for $x, y \in B_\omega^s$.

Proof. This follows by combining Proposition 3.3.2 and Lemma 3.3.5. Indeed, $\mathcal{C}^+ = \pi\sigma^{-L}\mathcal{B}$ with $\mathcal{B}$ given by (3.3.5). □

By ergodicity of ν^+ we find that for ν^+-almost all $\omega \in \Sigma_k^+$, the positive orbit under σ intersects $\mathcal{C}^+$ infinitely often. For such ω and for any $x, y \in (0,1)$, there is $n \geqslant 0$ so that $f_\omega^n(x)$ and $f_\omega^n(y)$ are contained in $B_{\sigma^n\omega}^s$. Thus (3.3.17) holds. □

The condition in Theorem 3.3.15 on minimality of the iterated function system on $\operatorname{supp}(m)$ is needed as the example with nonunique stationary measures in Section 3.4.3 makes clear. In contrast to Theorem 3.3.1, synchronization is shown for all $x, y \in (0,1)$.

3.4 Intermittency

As before, consider logistic maps

$$f_i(x) = \rho_i x(1-x),$$

$1 \leqslant i \leqslant k$, $0 < \rho_i \leqslant 4$, on $I = [0,1]$. We say that IFS $(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, displays intermittency if the following holds for any small neighborhood U of 0 and Lebesgue almost any $x \in (0,1)$: for ν^+-almost all $\omega \in \Sigma_k^+$,

(a) $f_\omega^n(x) \notin U$ for infinitely many n;

(b) $\lim_{N\to\infty} \frac{1}{N} |\{0 \leqslant n < N \,;\, f_\omega^n(x) \in U\}| = 1$.

Here, for a finite set S, we write $|S|$ for its cardinality. Compare also [13], where intermittency is studied in a context of interval diffeomorphisms and zero Lyapunov exponents. We establish intermittency in a set-up of iterated function systems generated by k logistic maps $f_1, \ldots, f_k$ that include the two logistic maps $f_{i_1}(x) = 2x(1-x)$ and $f_{i_2}(x) = 4x(1-x)$. The following theorem includes a condition on $L(0)$, the Lyapunov exponent at 0 given by (3.2.5). A result in this direction, focusing on null recurrence, is indicated in [8] for the case of the iterated function system with two maps $f_1(x) = 2x(1-x)$ and $f_2(x) = 4x(1-x)$.

Theorem 3.4.1. *Let IFS$(\mathbb{F})$, $\mathbb{F} = \{f_1, , \ldots, f_k\}$, be an iterated function system of logistic maps $f_i(x) = \rho_i x(1-x)$, taken with probabilities $p_i > 0$, with the following conditions.*

(i) There is i_1 with $f_{i_1}(x) = 2x(1-x)$;

(ii) There is i_2 with $f_{i_2}(x) = 4x(1-x)$;

(iii) $L(0) > 0$.

If $p_{i_1} > \frac{1}{2}$, then the delta measure at zero is the unique stationary measure. Moreover, IFS$(\mathbb{F})$ displays intermittency.

This theorem will be proved in Section 3.4.1. Other cases of intermittent time series are considered in Section 3.4.3. Contrasting with the above result is the following topological characterization.

Proposition 3.4.2. *Let IFS$(\mathbb{F})$, $\mathbb{F} = \{f_1, , \ldots, f_k\}$, be an iterated function system of logistic maps $f_i(x) = \rho_i x(1-x)$, taken with probabilities $p_i > 0$. Suppose there is i_2 with $f_{i_2}(x) = 4x(1-x)$. Then the skew product system $F^+ : \Sigma_k^+ \times I \to \Sigma_k^+ \times I$ is topologically mixing.*

Proof. Let U, V be open sets in $\Sigma_k^+ \times I$. Write $P : \Sigma_k^+ \times I \to \Sigma_k^+$ for the coordinate projection. Since σ on Σ_k^+ is an expanding map, $P(F^+)^{t_1}(U) = \Sigma_k^+$ for some t_1 large enough. The set U therefore contains a point (ω, x) with $\omega_i = i_2$ for all $i \geqslant t_1$. The map f_{i_2} has the expansion property that for any nontrivial interval $J \subset I$, there is an $n > 0$ so that $I \subset f_{i_2}^n(J)$. From this property of f_{i_2} one gets that some further iterate $(F^+)^{t_2}((F^+)^{t_1}(U))$ contains $\{\sigma^{t_1}\omega\} \times I = \{(i_2)^\infty\} \times I$, where $(i_2)^\infty$ stands for the sequence of only symbols i_2 in Σ_k^+. Again using that σ on Σ_k^+ is an expanding map, there exists $t > t_1 + t_2$ so that $(F^+)^t(U)$ covers $\Sigma_k^+ \times I$ and hence $(F^+)^n(U)$ intersects V for all $n \geqslant t$. $\square$

In more detail we will look at the iterated function system generated by just the two logistic maps

$$f_1(x) = 2x(1-x), \tag{3.4.1}$$
$$f_2(x) = 4x(1-x). \tag{3.4.2}$$

For the iterates, pick f_1 with probability p_1, $0 < p_1 < 1$, and f_2 with probability $p_2 = 1 - p_1$. Reminiscent of results for the Pomeau-Manneville map [24], we discuss σ-finite stationary measures in the following theorem. Here a σ-finite stationary measure is a σ-finite measure that satisfies the same identity that defines a stationary measure.

Theorem 3.4.3. *The iterated function system IFS$(\{f_1, f_2\})$ with f_1, f_2 given by* (3.4.1), (3.4.2)*, admits an absolutely continuous σ-finite stationary measure, which is not finite for $p_1 > 1/2$.*

We do not have a proof that this absolutely continuous σ-finite stationary measure is finite for $p_1 < 1/2$. The construction used to prove Theorem 3.4.3 gives the following, irrespective of the value of p_1. Denote Lebesgue measure on I by λ.

Proposition 3.4.4. *Consider the iterated function system IFS$(\{f_1, f_2\})$ with f_1, f_2 given by* (3.4.1), (3.4.2)*. For $\nu^+ \times \lambda$-almost all (ω, x), its orbit under F^+ lies dense in $\Sigma_2^+ \times I$.*

Before starting proofs we include a remark on the discontinuous dependence of stationary measures on parameters. Consider an iterated function system IFS$(\{f_1, \ldots, f_{k-1}\})$, with $f_{i_1}(x) = 2x(1-x)$ and $p_{i_1} > 1/2$. Assume none of the maps f_i, $1 \leqslant i < k$, equals $x \mapsto 4x(1-x)$. If further $L(0) > 0$, then by Proposition 3.3.7 there is a stationary measure m with $m(\{0\}) = 0$. Now include the map $f_k(x) = 4x(1-x)$ with probability $p_k = \varepsilon$, multiplying the other probabilities p_i with $1 - \varepsilon$ to ensure that the sum of probabilities stays 1. For ε small, by Theorem 3.4.1 the only stationary measure for IFS$(\mathbb{F})$, $\mathbb{F} = \{f_1, \ldots, f_k\}$, is the delta measure δ_0 at 0. We conclude that the set of stationary measures for IFS$(\mathbb{F})$ changes discontinuously in ε, at $\varepsilon = 0$.

3.4.1 Proof of Theorem 3.4.1

Our proof of Theorem 3.4.1 is a direct study of time series: in typical time series one expects a frequent occurrence of compositions $f_{i_2} \circ f_{i_1}^{\ell}$ with large $\ell > 0$. In such a composition, a point which is not too close to 0 or 1 is first mapped by $f_{i_1}^{\ell}$ to a point very close to $1/2$ (since $1/2$ is a superstable fixed point for f_{i_1}) and then by f_{i_2} to a point very close to 1. The next iterates bring the point first very close to 0 after which a very large number of iterates is needed to map the point outside a neighborhood of 0. In the proof we supply the estimates making this explicit.

Proof of Theorem 3.4.1. We claim that for all $x \in I$ there is a set $\Omega_x \subset \Sigma_k^+$ of full ν^+ measure so that for $\omega \in \Omega_x$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{f_\omega^n(x)} = \delta_0, \tag{3.4.3}$$

with convergence in the weak star topology.

It follows from this claim that δ_0 is the unique stationary measure. To prove uniqueness: suppose there is another ergodic stationary measure m. Then $\nu^+ \times m$ is an invariant measure for F^+. Recall from (3.2.2) that for $\nu^+ \times m$-almost every (ω, x), we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{f_\omega^n(x)} = m. \tag{3.4.4}$$

By Fubini's theorem there is a subset of I of full m-measure, so that in any $\Sigma_k^+ \times \{x\}$ with x from this subset, there is a set of full ν^+-measure for which (3.4.4) holds. This however contradicts (3.4.3), since that applies to all x in I.

For $\varepsilon > 0$ write

$$U_\varepsilon = [0, \varepsilon) \cup (1-\varepsilon, 1].$$

We prove (3.4.3) by establishing that for all small $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} |\{0 \leqslant n < N \ ; \ f_\omega^n(x) \notin U_\varepsilon\}| = 0. \tag{3.4.5}$$

This is equivalent to (3.4.3).

We will however first show that for Lebesgue almost all $x \in (0,1)$, ν^+-almost all $\omega \in \Sigma_k^+$ and ε small enough, $f_\omega^n(x) \notin U_\varepsilon$ for infinitely many $n \in \mathbb{N}$. It is a consequence of the following lemma. This lemma is redundant in case $f_i'(0) > 1$ for all $1 \leqslant i \leqslant k$, for instance the iterated function system with only $x \mapsto 2x(1-x)$ and $x \mapsto 4x(1-x)$ included.

Lemma 3.4.5. *For every small $\varepsilon > 0$ and ν^+-almost all $\omega \in \Sigma_k^+$, for any $x \in (0, \varepsilon]$ there is $N > 0$ with $f_\omega^N(x) > \varepsilon$.*

Proof. Recall the assumption $L(0) = \sum_{i=1}^k p_i \ln(f_i'(0)) > 0$. For every $\delta > 0$ there is $\varepsilon_0 > 0$ such that for every $0 < x \leqslant \varepsilon_0$ and $1 \leqslant i \leqslant k$,

$$|f_i'(x) - f_i'(0)| < \delta. \tag{3.4.6}$$

By Birkhoff's ergodic theorem, for ν^+-almost all $\omega \in \Sigma_k^+$,

$$\lim_{N\to\infty} \frac{1}{N} \left|\{0 \leqslant n < N \ ; \ \sigma^n\omega \in C_i^0\}\right| = p_i, \ \forall\, 1 \leqslant i \leqslant k. \tag{3.4.7}$$

For $N \in \mathbb{N}$ and $1 \leqslant i \leqslant k$, denote

$$q_i(N) = \frac{1}{N} \left|\{0 \leqslant n < N \ ; \ \sigma^n\omega \in C_i^0\}\right|.$$

For given ω for which (3.4.7) holds, we have that for every $\delta > 0$ there exists $N_0 > 0$ such that for every $N > N_0$,

$$|q_i(N) - p_i| < \delta, \ \forall\, 1 \leqslant i \leqslant k. \tag{3.4.8}$$

Take $\delta > 0$ small so that $p_i - \delta > 0$, $f_i'(0) - \delta > 0$ for every $1 \leqslant i \leqslant k$ and

$$\sum_{i=1}^{k} (p_i - \delta) \ln(f_i'(0) - \delta) > 0.$$

Take $\varepsilon_0 > 0$ and $N_0 \in \mathbb{N}$ large such that (3.4.6) and (3.4.8) are satisfied. Take $0 < \varepsilon \leqslant \varepsilon_0$ and $x \in (0, \varepsilon]$. Also take ω for which (3.4.7) holds and suppose that for every $N \geqslant 1$, $x, \ldots, f_\omega^N(x) \in (0, \varepsilon]$. Calculate

$$\begin{aligned} \frac{1}{N} \ln \left| f'_{\omega_{N-1}}(f_\omega^{N-1}(x)) \cdots f'_{\omega_0}(x) \right| &= \frac{1}{N} \sum_{j=0}^{N-1} \ln |f'_{\sigma^j \omega}(f_\omega^j(x))| \\ &> \sum_{i=1}^{k} q_i(N) \ln(f_i'(0) - \delta) \\ &> \sum_{i=1}^{k} (p_i - \delta) \ln(f_i'(0) - \delta), \end{aligned}$$

for every $N > N_0$. Since the right hand side is positive, it follows that for $N > N_0$ the map f_ω^N is expanding at x, which is a contradiction. So $f_\omega^N(x) > \varepsilon$ for some $N \geqslant 1$. □

Fix $\varepsilon > 0$ small for which Lemma 3.4.5 holds. In order to obtain (3.4.5), we need information on the numbers of consecutive iterates spend in the different sets U_ε and $I \setminus U_\varepsilon$. We focus on orbit pieces that start at a point in $I \setminus U_\varepsilon$, contain a sufficiently large number of iterates of f_{i_1} (bringing the point close to 1/2) and then one iterate of f_{i_2} (bringing the point close to 1). The following lemma provides necessary estimates.

Lemma 3.4.6. *Assume all conditions of Theorem 3.4.1 hold. There exist constants $L, T > 0$ so that*

(i) For every $y \in I \setminus U_\varepsilon$, $y' = f_{i_2} \circ f_{i_1}^\ell(y) \in U_\varepsilon$ for every $\ell \geqslant L$.

(ii) Let $y \in I \setminus U_\varepsilon$ and $y' \in U_\varepsilon$ as in item (i). For $\eta \in \Sigma_k^+$, let $h = h(\eta, y) > 0$ be the smallest integer with $f_\eta^h(y') \in I \setminus U_\varepsilon$. Then $h \geqslant T2^\ell$.

Proof. With $c = 2$ we have that on small closed neighborhoods V of the critical point,

$$|f_{i_1}(y) - \frac{1}{2}| \leqslant c|y - \frac{1}{2}|^2, \tag{3.4.9}$$

for every $y \in V$. Write $V = [\frac{1}{2} - \kappa, \frac{1}{2} + \kappa]$ for $\kappa > 0$ and take V small so that $c\kappa < 1$. By (3.4.9) for every $\ell \in \mathbb{N}$ and $y \in V$

$$|f_{i_1}^\ell(y) - 1/2| \leqslant c^{2^\ell - 1}|y - 1/2|^{2^\ell},$$

and
$$|f_{i_2} \circ f_{i_1}^{\ell}(y) - 1| \leqslant 4c^{2(2^{\ell}-1)}|y - 1/2|^{2^{\ell+1}}.$$

Therefore, for every $y \in V$ the distance of $f_{i_2} \circ f_{i_1}^{\ell}(y)$ to 1 is smaller than $(\frac{2}{c})^2(c\kappa)^{2^{\ell+1}}$. Since $c\kappa < 1$, there is $L > 0$ such that $(\frac{2}{c})^2(c\kappa)^{2^{\ell+1}} < \varepsilon$ for every $\ell \geqslant L$. This gives item (i) for $y \in V$. Since $f_i(x) \leqslant 4x$ for all $1 \leqslant i \leqslant k$, we have $f_{\eta}^{h}(x) \leqslant 4^h x$. The number h of iterates needed to map $f_{i_2} \circ f_{i_1}^{\ell}(y)$ outside of U_ε satisfies $4^h(\frac{2}{c})^2(c\kappa)^{2^{\ell+1}} \geq \varepsilon$ and thus $h \geq T2^{\ell}$, for some $T > 0$ (depending only on ε). This gives item (ii) for $y \in V$. $\square$

Write
$$\mathbb{A} = \Sigma_k^+ \times (I \setminus U_\varepsilon)$$
and let $F_{\mathbb{A}}^+ : \mathbb{A} \to \mathbb{A}$ be the first return map on $\mathbb{A}$.

Note that $F_{\mathbb{A}}^+$ is not defined everywhere, in particular not on $C_{i_2}^0 \times \{1/2\}$. The set $\mathbb{H} = \mathbb{A} \cap \bigcup_{n \geqslant 0}(F^+)^{-n}(C_{i_2}^0 \times \{1/2\})$ is a countable union of segments $C_{\upsilon_0,\ldots,\upsilon_{n-1}}^{0,\ldots,n-1} \times \{y\}$ with $y \in I \setminus U_\varepsilon$ and $f_{\upsilon}^n(y) = 1/2$. In particular, $\mathbb{H}$ involves a countable set Z of points $y \in \mathbb{A}$. Note that (3.4.5) holds for $(\omega, x) \in \mathbb{H}$. We find it convenient to artificially alter $F_{\mathbb{A}}^+$ on $C_{i_2}^0 \times \{1/2\}$. For this purpose, for $(\omega, x) \in \mathbb{H}$, we define $F_{\mathbb{A}}^+(\omega, x) = (\sigma\omega, y)$ for some $y \notin Z$.

We claim that, after this alteration, there is a set $\Omega \subset \Sigma_k^+$ with $\nu^+(\Omega) = 0$ so that $F_{\mathbb{A}}^+$ is defined on $\Sigma_k^+ \setminus \Omega \times (0,1)$. By Lemma 3.4.5, $\nu^+(\Omega_0) = 0$ for
$$\Omega_0 = \{\omega \in \Sigma_k^+ \; ; \; \exists 0 < x < \varepsilon, \forall n \in \mathbb{N}, f_\omega^n(x) < \varepsilon\}.$$

Then also $\nu^+(\Omega) = 0$ with $\Omega = \cup_{n \geqslant 0}\sigma^{-n}(\Omega_0)$. For $(\omega, x) \in (\Sigma_k^+ \setminus \Omega) \times (0,1)$, $(F^+)^n(\omega, x)$ is defined for all $n \in \mathbb{N}$ and $f_\omega^n(x) \in I \setminus U_\varepsilon$ infinitely often.

Let
$$\mathbb{A}_1 = \{(\eta, y) \in \mathbb{A} \; ; \; (\eta, y) \in C_{i_1^{\ell}, i_2}^{0,\ldots,\ell} \times I \setminus U_\varepsilon, \; \ell \geqslant L\},$$

corresponding to points in $I \setminus U_\varepsilon$ that are first iterated by $f_{i_2} \circ f_{i_1}^{\ell}$ for sufficiently large values of ℓ. Write
$$\mathbb{A}_2 = \mathbb{A} \setminus \mathbb{A}_1.$$

Consider $y' \in U_\varepsilon$ obtained from item (i) of Lemma 3.4.6, an image of some $y \in I \setminus U_\varepsilon$. Remember that $h(\eta, y)$ is the minimal positive integer that $f_\eta^{h(\eta,y)}(y') \in I \setminus U_\varepsilon$. For a time series $\{(\omega, x), F_{\mathbb{A}}^+(\omega, x), \ldots\}$ for $F_{\mathbb{A}}^+$, we distinguish different orbit pieces. A point $(\eta, y) = (F_{\mathbb{A}}^+)^n(\omega, x)$ in $\mathbb{A}_1$ for $\ell \geqslant L$ yields an orbit piece
$$(\eta, y), (\sigma\eta, f_\eta(y)), \ldots, (\sigma^{\ell}\eta, f_\eta^{\ell}(y)), \ldots, (\sigma^{\ell+h(\eta,y)}\eta, f_\eta^{\ell+h(\eta,y)}(y))$$

for F^+. The first $\ell + 1$ points (η, y) with $\eta = (i_1 \ldots i_1 \, i_2 \ldots)$ up to $(\sigma^{\ell}\eta, f_\eta^{\ell}(y))$ with $\sigma^{\ell}\eta = (i_2 \ldots)$ are in $\mathbb{A}$, as is the final point $(\sigma^{\ell+h(\eta,y)}\eta, f_\eta^{\ell+h(\eta,y)}(y))$. The time series for $F_{\mathbb{A}}^+$ is union of all F^+ orbit pieces starting in $\mathbb{A}$, where the final point of an orbit piece is the initial point of the next orbit piece.

Consider an orbit piece of length N for F^+ with an initial point $(\omega, x) \in \mathbb{A}$ for N large. Let M be the length of the corresponding orbit piece for $F^+_{\mathbb{A}}$. Denote

$$\alpha_1 = \left|\{0 \leqslant n < M \ ; \ (F^+_{\mathbb{A}})^n(\omega, x) \in \mathbb{A}_1\}\right|,$$
$$\alpha_2 = \left|\{0 \leqslant n < M \ ; \ (F^+_{\mathbb{A}})^n(\omega, x) \in \mathbb{A}_2\}\right|.$$

We have $M = \alpha_1 + \alpha_2$. The probability that points in $I \setminus U_\varepsilon$ are first iterated by $f_{i_2} \circ f^\ell_{i_1}$ for some $\ell \geqslant L$ is $\nu^+(C^{0,\dots,\ell}_{i_1^\ell, i_2}) = p^\ell_{i_1} p_{i_2}$. Denote $\pi_0(\omega, x) = \omega_0$ and consider the sequence of random variables

$$\eta_n = \pi_0(F^+_{\mathbb{A}})^n(\omega, x).$$

We formulate the following, perhaps intuitively obvious, statement.

Lemma 3.4.7. *The sequence of stochastic variables η_n are independently distributed with identical distribution, in which values $1, \dots, k$ have probabilities $p_1, \dots,$ p_k.*

Proof. It is clear that each η_n is distributed with probabilities $p_1, \dots, p_k$ for the values $1, \dots, k$, because each η_n equals some ω_i. For instance the probability $P(\eta_i = k)$ equals the sum $\sum_{s \geqslant 0} P(\eta_i = \omega_s, \omega_s = k)$. Since the ω_m's are independent and for ν^+-almost all $\omega \in \Sigma^+_k$, there are infinitely many η_n's, this probability equals $P(\omega_s = k)$ times the total probability of the set of sequences that give $\eta_i = \omega_s$ for some $s \geqslant 0$. This total probability is one, so that $P(\eta_i = k) = p_k$.

For independence we must show that the probability $P(\eta_{i_1} = k_1, \dots, \eta_{i_h} = k_h)$, $h \geqslant 2$, equals the product of probabilities $P(\eta_{i_j} = k_j)$,$1 \leqslant j \leqslant h$. For simplicity we consider $h = 2$, a higher number of events goes similarly. We have $P(\eta_i = k, \eta_j = l) = \sum_{0 \leqslant s < t} P(\eta_i = \omega_s, \eta_j = \omega_t, \omega_s = k, \omega_t = l)$. Again by independence of the ω_m's and since for ν^+-almost all $\omega \in \Sigma^+_k$, there are infinitely many η_n's, this probability equals $p_k p_l$. □

By Lemma 3.4.7 and Kolmogorov's strong law of large numbers, see e.g. [27, Chapter IV, §3], for ν^+-almost all $\omega \in \Sigma^+_k$,

$$\lim_{M \to \infty} \frac{\alpha_1}{M} = p^\ell_{i_1} p_{i_2}. \tag{3.4.10}$$

By item (ii) we know that for every $\ell \geqslant L$ and M large, $N \geqslant T 2^\ell \alpha_1 + M$. From (3.4.10) we see that that $T 2^\ell \frac{\alpha_1}{M}$ is arbitrary large if $2p_{i_1} > 1$ and ℓ, M large. Thus we can calculate

$$\lim_{N \to \infty} \frac{1}{N} \left|\{0 \leqslant n < N \ ; \ f^n_\omega(x) \notin U_\varepsilon\}\right| = \lim_{M \to \infty} \frac{M}{N} \leqslant \lim_{M \to \infty} \frac{M}{T 2^\ell \alpha_1 + M},$$

where the last value is arbitrary close to 0 for ℓ large. This proves (3.4.5) and the theorem is proved. □

3.4.2 Proof of Theorem 3.4.3

In this section we prove Theorem 3.4.3 and obtain Proposition 3.4.4 as a corollary. So we continue with the iterated function system generated by the two maps $f_1(x) = 2x(1-x)$ and $f_2(x) = 4x(1-x)$.

To prove Theorem 3.4.3 we adapt the line of thought that is used in the study of invariant measures for interval maps such as Pomeau-Manneville maps [24] or Misiurewicz maps [22, Chapter V, Section 3]. Here one proceeds through the construction of a first return map on a subinterval. The first return map is a Markov map for which one constructs an invariant measure by finding its density as the fixed point of a Perron-Frobenius operator. This then provides an invariant measure for the original system, whose finiteness depends on integrability of the return times.

Proof of Theorem 3.4.3. Take the two critical points of f_2^3 in $(0,1/2)$ that are closest to zero. Let J be the open interval between these two points. Note that J is of the form $(r, f_2(r))$ with $0 < r < f_2(r) < 1/2$. Define

$$\mathbb{B} = C^{0,1,2}_{2,2,2} \times J.$$

The boundary $\partial\mathbb{B}$ of $\mathbb{B}$ equals $C^{0,1,2}_{2,2,2} \times \{r, f_2(r)\}$. It is immediate that the following lemma holds.

Lemma 3.4.8.

$$(F^+)^n(\partial\mathbb{B}) \cap \mathbb{B} = \emptyset,$$

for all $n \geqslant 0$.

Consider the first return map $F^+_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$ associated to F^+. That is,

$$F^+_{\mathbb{B}}(\omega, x) = (\sigma^s\omega, f^s_\omega(x)),$$

with $s = s(\omega, x)$ the smallest positive natural number with $\sigma^s\omega \in C^{0,1,2}_{2,2,2}$ and $f^s_\omega(x) \in J$. Each map f^s_ω has its set of critical values equal to either $\{0, 1/2\}$ (if $\omega_{s-1} = 1$) or $\{0,1\}$ (if $\omega_{s-1} = 2$). The domains on which $F^+_{\mathbb{B}}$ is continuous, that is on which $s(\omega,x)$ is constant, are products of cylinders $C^{0,\ldots,s(\omega,x)-1}_{\eta_0,\ldots,\eta_{s(\omega,x)-1}}$ with $\eta_0, \eta_1, \eta_2 = 2$ and open intervals in J. Let $\mathbb{P} = \{\mathbb{P}_i\}$ be the countable partition consisting of these domains. Thus each $\mathbb{P}_i$ is a product of a cylinder $C_i = C^{0,\ldots,s_i-1}_{\eta_0,\ldots,\eta_{s_i-1}}$ and an open interval K_i. . By Lemma 3.4.8,

$$F^+_{\mathbb{B}}(\mathbb{P}_i) = \mathbb{B} \tag{3.4.11}$$

for each $i \geqslant 1$. Write

$$\mathbb{E} = \cup_i \mathbb{P}_i$$

for the domain of $F^+_{\mathbb{B}}$.

Take a reference probability measure $\nu^+ \times \lambda$ on $\Sigma_2^+ \times J$, where λ stands for a multiple of Lebesgue measure. The next step is to prove that $\mathbb{E}$ has full measure in $\mathbb{B}$ (for this reference measure) as stipulated by Lemma 3.4.9 below.

Given a measurable set $\mathbb{S} \subset \Sigma_2^+ \times I$, a property is said to hold for $\nu^+ \times \lambda$-almost all $(\omega, x) \in \mathbb{S}$, if the set of points in $\mathbb{S}$ for which it does not hold has zero measure. We call (ω, x) a fiber density point of $\mathbb{S}$ if it is a Lebesgue density point in the fiber $\{\omega\} \times I$:

$$\lim_{D \longrightarrow \{x\}} \frac{\lambda(\mathbb{S} \cap (\{\omega\} \times I) \cap D)}{\lambda(D)} = 1,$$

where the limit is over decreasing intervals $D \ni x$.

In the next lemma we will also use the notion of horizontal density points in sets $\left(\Sigma_2^+ \times \{x\}\right) \cap \mathbb{S}$, where the notion is defined through the isomorphism of the shift map on Σ_2^+ and a piecewise expanding map on I. This isomorphism arises as follows. Consider the expanding interval map $g : I \to I$ that expands intervals $I_1 = [0, p_1]$, $I_2 = [p_1, 1]$ by factors $\frac{1}{p_1}$, $\frac{1}{p_2}$ respectively: writing $p_0 = 0$,

$$g(y) = \frac{y - p_{i-1}}{p_i},\ y \in I_i.$$

Note that g preserves Lebesgue measure λ. An itinerary $\omega \in \Sigma_2^+$ corresponds to a point $y \in I$ via

$$y = \sum_{i=0}^{\infty} l_{\omega(i)} \prod_{j=0}^{i-1} p_{\omega(j)}.$$

This formula defines a map $h : \Sigma_2^+ \to I$ that provides a topological semi-conjugacy $g \circ h = h \circ \sigma$. Now h defines an isomorphism because the Bernoulli measure on Σ_2^+ is pushed forward to Lebesgue measure on I by h. Through this isomorphism of Σ_2^+ with I, we speak of horizontal density points of $\mathbb{S}$ in $\Sigma_2^+ \times \{x\}$.

Lemma 3.4.9. $\nu^+ \times \lambda(\mathbb{B} \setminus \mathbb{E}) = 0$.

Proof. Suppose by contradiction that there is a set $\mathbb{S}$ of points $(\omega, x) \in \mathbb{B}$, of positive measure, for which the orbit stays outside of $\mathbb{B}$. By Fubini's theorem and the Lebesgue density theorem, $\nu^+ \times \lambda$-almost all points $(\omega, x) \in \mathbb{S}$ are horizontal density points of $\mathbb{S} \cap (C^{0,1,2}_{2,2,2} \times \{x\})$.

We first claim that for $\nu^+ \times \lambda$-almost all $(\omega, x) \in \mathbb{B}$, $f^n_\omega(x) \in J$ infinitely often. Recall from item (i) of Lemma 3.4.6 that for ν^+-almost all $\omega \in \Sigma_2^+$ and $\ell \geqslant L$, we have $\sigma^i \omega \in C^{0,\ldots,\ell}_{1,\ldots,1,2}$ infinitely often. For such ω and $x \in I$ there can not be $m \in \mathbb{N}$ so that $f^i_\omega(x) \in I \setminus U_\varepsilon$ for all $i > m$. Thus for such ω we have $f^n_\omega(x) \in U_\varepsilon$ for infinitely many values of n. Remove the set $\cup_{n \in \mathbb{N}} (F^+)^{-n}(C^0_2 \times \{1/2\})$ of points that are mapped onto 0 by some iterate. This set has zero measure for $\nu^+ \times \lambda$. The claim now follows since each point in $(0, \varepsilon)$ will pass through J under iteration.

By altering $\mathbb{S}$ we may thus assume that for every point $(\omega, x) \in \mathbb{S}$, $f^n_\omega(x) \in J$ infinitely often. Now take $(\omega, x) \in \mathbb{S}$ such that (ω, x) is a horizontal density point of $\mathbb{S}$ in $C^{0,1,2}_{2,2,2} \times \{x\}$. Suppose $f^{n(j)}_\omega(x) \in J$ for infinitely many positive values $n(j)$. Let $\chi_{n(j)}$ be the smallest neighborhood of ω with $\sigma^{n(j)}(\chi_{n(j)}) = \Sigma_2^+$. Observe that for every $\eta \in \chi_{n(j)}$, we have $f_\eta^{n(j)}(x) = f_\omega^{n(j)}(x) \in J$. Observe that for every j, $\chi_{n(j+1)} \subset \chi_{n(j)}$ and $\chi_{n(j)} \to \{\omega\}$ as $j \to \infty$.

Since ω is a horizontal density point of $\mathbb{S} \cap (C^{0,1,2}_{2,2,2} \times \{x\})$, for every j there exists a set $\mathcal{D}_{n(j)} \subset \chi_{n(j)}$ of positive measure such that $\mathcal{D}_{n(j)} \times \{x\} \subset \mathbb{S}$ and

$$\lim_{j\to\infty} \frac{\nu^+(\mathcal{D}_{n(j)})}{\nu^+(\chi_{n(j)})} = 1. \tag{3.4.12}$$

Since $\sigma^{n(j)}(\chi_{n(j)}) = \Sigma_2^+$ there exists a set $\mathcal{E}_{n(j)} \subset \chi_{n(j)}$ such that $\sigma^{n(j)}(\mathcal{E}_{n(j)}) = C^{0,1,2}_{2,2,2}$. For every j and $\eta \in \mathcal{E}_{n(j)}$, $F^{n(j)}(\eta, x) \in \mathbb{B}$. For every j, $\frac{\nu^+(\mathcal{E}_{n(j)})}{\nu^+(\chi_{n(j)})}$ is the positive constant $\frac{\nu^+(C^{0,1,2}_{2,2,2})}{\nu^+(\Sigma_2^+)}$, while $\mathcal{E}_{n(j)} \cap \mathcal{D}_{n(j)} = \emptyset$. This is a contradiction with (3.4.12). $\square$

We find an invariant measure for $F_\mathbb{B}^+$ by pushing forward $\nu^+ \times \lambda$ under iterates of $F_\mathbb{B}^+$;

$$(F_\mathbb{B}^+)^n\,(\nu^+ \times \lambda)(U) = \sum_{i=1}^\infty \left(F_\mathbb{B}^+|_{\mathbb{P}_i}\right)^{-1}(U)$$

for Borel sets $U \subset \mathbb{B}$.

Since $F_\mathbb{B}^+$ restricted to a partition element is a product map, also $(F_\mathbb{B}^+)^n\,(\nu^+ \times \lambda)$ is a product measure. Because of (3.4.11) and Lemma 3.4.9,

$$(F_\mathbb{B}^+)^n\,(\nu^+ \times \lambda)(\mathbb{B}) = \sum_{i=1}^\infty (\nu^+ \times \lambda)(\mathbb{P}_i) = 1.$$

So

$$(F_\mathbb{B}^+)^n\,(\nu^+ \times \lambda) = \nu^+ \times m_n$$

for some finite measure m_n on I, with $\nu^+ \times m_n$ a probability measure. In order to investigate the densities of m_n, define a Perron-Frobenius operator $\mathcal{P} : L^1(J, \lambda) \to L^1(J, \lambda)$ by

$$\mathcal{P}\varphi(x) = \sum_{i=1}^\infty \nu^+(C_i) \frac{\varphi(g_i^{-1}(x))}{|g_i'(g_i^{-1}(x))|}.$$

Here $F_\mathbb{B}^+|_{\mathbb{P}_i} = \sigma^{s(i)} \times g_i$. Now observe that, if $\varphi_n \in L^1(J, \lambda)$ denotes the density $dm_n/d\lambda$ of m_n, we get

$$\varphi_n = \mathcal{P}^n 1. \tag{3.4.13}$$

With this formula in place, we obtain the following result formulated as a lemma.

Lemma 3.4.10. *$F_\mathbb{B}^+$ admits an invariant measure $\nu^+ \times m_\mathbb{B}$ where $m_\mathbb{B}$ is absolutely continuous with respect to Lebesgue measure. Moreover, there are positive constants c_1, c_2 so that*

$$c_1 < \frac{dm_\mathbb{B}}{d\lambda} < c_2.$$

Proof. The proof of the folklore theorem as in [22, Chapter V, Theorem 2.2] studies iterates (3.4.13) for a Perron-Frobenius operator obtained from a Markov map as defined in [22, Chapter V, Section 2]. We can follow the proof of the folklore theorem [22, Chapter V, Theorem 2.2] verbatim to obtain the result.

Indeed, the essential uniform bounded distortion estimates for branches of Markov maps (see Koebe's principle [22, Chapter IV, Theorem 1.2]) also hold for the branches of $F^+_{\mathbb{B}}$ (the maps g_i) and for branches of iterates $(F^+_{\mathbb{B}})^n$, because each g_i has negative Schwarzian derivative. □

Finally, an invariant measure μ of F^+ is as usual obtained from $\nu^+ \times m_{\mathbb{B}}$ by

$$\mu = \sum_{j=1}^{\infty} \sum_{i=0}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right)$$

Invariance follows from

$$\begin{aligned}
F^+ \mu &= F^+ \sum_{j=1}^{\infty} \sum_{i=0}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{s(j)} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) + \sum_{j=1}^{\infty} (F^+)^{s(j)} \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) + F^+_{\mathbb{B}} \left(\nu^+ \times m_{\mathbb{B}}\right) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{s(j)-1} (F^+)^i \left(\nu^+ \times m_{\mathbb{B}}|_{C_j \times K_j}\right) + \nu^+ \times m_{\mathbb{B}} \\
&= \mu.
\end{aligned}$$

Because

$$\mu(\mathbb{B}) = \mu(C^{0,1,2}_{2,2,2} \times J) = \sum_{j=1}^{\infty} \sum_{i=0}^{s(j)-1} (\nu^+ \times m_{\mathbb{B}})(C_j \times K_j) = \sum_{j=1}^{\infty} s(j)\nu^+(C_j) m_{\mathbb{B}}(K_j),$$

μ is finite if the last sum is finite. From Theorem 3.4.1 it follows that μ is not finite if $p_1 > 1/2$. □

Proposition 3.4.4 is obtained as a corollary to Theorem 3.4.3.

Proof of Proposition 3.4.4. The statement is a consequence of the invariant probability measure $\nu^+ \times m_{\mathbb{B}}$ for the first return map occurring in the above proof of

Theorem 3.4.3. This is an ergodic measure (see [22, Chapter V, Theorem 2.2]). Therefore by Birkhoff's ergodic theorem and absolute continuity of $m_{\mathbb{B}}$, for $\nu^+ \times \lambda$-almost all initial points, its orbit under $F^+_{\mathbb{B}}$ intersects any open set in $\mathbb{B}$ infinitely often. This then also holds for the orbit under F^+ and any open set in $\Sigma_2^+ \times I$ by appealing to Proposition 3.4.2. □

3.4.3 Other cases of intermittency

In the previous sections we treated intermittency in iterated function systems generated by a logistic map with a superstable fixed point and $x \mapsto 4x(1-x)$. In this section we address some other cases leading to intermittent time series. We will not provide detailed proofs.

The arguments to prove Theorem 3.4.1 can be used to prove intermittency involving superstable periodic orbits of higher period. We provide a statement in the following result. The left frame of Figure 3.3 illustrates a time series, where we note that Theorem 3.4.11 does formally not cover the chosen iterated function system. We remark that Proposition 3.4.2 applies under the assumptions of the following theorem.

Theorem 3.4.11. *Let IFS* $(\mathbb{F})$, $\mathbb{F} = \{f_1, , \ldots, f_k\}$, *be an iterated function system of logistic maps* $f_i(x) = \rho_i x(1-x)$, *taken with probabilities* $p_i > 0$, *with the following conditions.*

(i) *There is* i_1 *with* f_{i_1} *possessing a superstable periodic orbit of period* t: $f^t_{i_1}(1/2)$ $= 1/2$;

(ii) *There is* i_2 *with* $f_{i_2}(x) = 4x(1-x)$;

(iii) *There is* i_3 *with* $f_{i_3}(x) = \rho_{i_3} x(1-x)$ *with* $1 < \rho_{i_3} < 3$;

(iv) *There is* i_4 *with* $f_{i_4}(x) = \rho_{i_4} x(1-x)$ *with* $0 < \rho_{i_4} < 1$;

(v) $L(0) > 0$.

If $p_{i_1} > \sqrt[t]{\frac{1}{2}}$, *then the delta measure at zero is the unique stationary measure. Moreover, IFS* $(\mathbb{F})$ *displays intermittency.*

Sketch of proof. The arguments to prove Theorem 3.4.1 can be largely followed. Lemma 3.4.6 is replaced by the following.

Lemma 3.4.12. *Assume all conditions of Theorem 3.4.1 hold. There exist constants* $L, T > 0$ *and a finite word* τ *of symbols in* $\{1, \ldots, k\}$ *such that the following items hold:*

(i) *For every* $y \in I \setminus U_\varepsilon$, $y' = f_{i_2} \circ f^{t\ell}_{i_1} \circ f_\tau(y) \in U_\varepsilon$ *for every* $\ell \geqslant L$.

(ii) *Let $y \in I \setminus U_\varepsilon$ and $y' \in U_\varepsilon$ obtained from item (i). Assume that for $\eta \in \Sigma_k^+$, $h = h(\eta, y) > 0$ is the smallest integer such that $f_\eta^h(y') \in I \setminus U_\varepsilon$. Then $h \geqslant T2^\ell$.*

Proof. By Proposition 3.3.13 there is $K \in \mathbb{N}$ and $\tau_0, \dots, \tau_{K-1}$ so that $f_\tau^K(I \setminus U_\varepsilon) \subset W_{loc}^s(1/2)$, the immediate basin of $1/2$ for f_{i_1} (the immediate basin of $1/2$ is the maximal interval containing $1/2$ that is in the basin of attraction of $1/2$ for $f_{i_1}^t$). The reasoning of Lemma 3.4.6 can now be followed almost verbatim with $f_{i_1}^t$ replacing f_{i_1}. □

Equation (3.4.10) is replaced by

$$\lim_{M \to \infty} \frac{\alpha_1}{M} = p(\tau) p_{i_1}^{t\ell} p_{i_2},$$

for ν^+-almost all $\omega \in \Sigma_k^+$. Here $p(\tau) = \nu^+(C_{\tau_0, \dots, \tau_{K-1}}^{0, \dots, K-1})$. The rest of the reasoning can be followed to conclude the proof. □

Now we consider a case of intermittency near an invariant set with zero Lyapunov exponents, akin to [4, 13]. Reference [3] provides examples of nonunique stationary measures that assign measure 0 to $\{0\}$. Indeed, taking two logistic maps

$$\begin{aligned} f_1(x) &= \rho_1 x(1-x),\ \rho_1 \in (2,4), \\ f_2(x) &= \rho_2 x(1-x),\ \rho_2 = \frac{\rho_1}{\rho_1 - 1} \in \left(\frac{4}{3}, 2\right), \end{aligned}$$

we find that

$$S = \{1/\rho_1, 1 - 1/\rho_1\} = \{1 - 1/\rho_2, 1/\rho_2\}$$

is an invariant set for IFS $(\{f_1, f_2\})$. These points are the fixed point $1 - 1/\rho_1$ of f_1 and the fixed point $1 - 1/\rho_2$ of f_2, see Figure 3.4. If f_1 and f_2 are picked with probabilities p_1 and p_2, then the measure m given by

$$m(\{1 - 1/\rho_1\}) = p_1,\ m(\{1 - 1/\rho_2\}) = p_2 \tag{3.4.14}$$

is a stationary measure supported on S. The Lyapunov exponent with respect to m equals

$$\begin{aligned} L &= \int_I p_1 \ln |f_1'(x)| + p_2 \ln |f_2'(x)|\, dm(x) \\ &= p_2 \left[p_1 \ln |f_1'(1/\rho_1)| + p_2 \ln |f_2'(1/\rho_1)|\right] + p_1 \left[p_1 \ln |f_1'(1/\rho_2)| + p_2 \ln |f_2'(1/\rho_2)|\right] \\ &= p_1 \ln(f_1'(1/\rho_1)) + p_2 \ln(f_2'(1/\rho_1)) \\ &= p_1 \ln(\rho_1(1 - 2/\rho_1)) + p_2 \ln(\rho_2(1 - 2/\rho_1)) \end{aligned} \tag{3.4.15}$$

and is positive for $\rho_1 > 3$ and p_2 small enough. Note that

$$\operatorname{supp}(m) \subset [f_2 \circ f_1(1/2), f_1(1/2)]$$

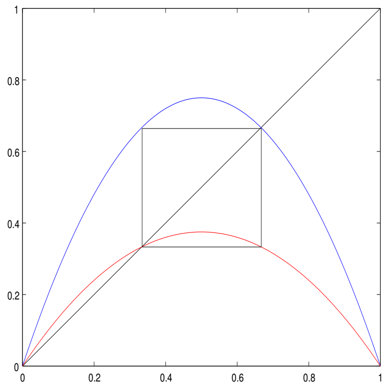

Figure 3.4: *The graphs of two logistic maps, for which the corresponding iterated function system has an invariant set S consisting of the two positive fixed points.*

for any stationary measure m with support in $(0,1)$. In [3] it is shown that if in addition to the positivity of the Lyapunov exponent (3.4.15), one assumes that $f_1^{-1}(1/\rho_1)$ is disjoint from $[f_2 \circ f_1(1/2), f_1(1/2)]$, there exists another stationary measure that also assigns measure 0 to $\{0\}$.

The following theorem finds intermittency near the invariant set S, meaning the following: for any small neighborhood U of S and any $x \in (0,1) \setminus S$, for ν^+-almost all $\omega \in \Sigma_2^+$,

(a) $f_\omega^n(x) \notin U$ for infinitely many n;

(b) $\lim_{N\to\infty} \frac{1}{N} |\{0 \leqslant n < N\ ;\ f_\omega^n(x) \in U\}| = 1$.

The right frame of Figure 3.3 illustrates a time series.

Theorem 3.4.13. *Consider the above families of logistic maps f_1 and f_2, with $3 < \rho_1 < 1 + \sqrt{5}$. Take f_1 with probability p_1 and f_2 with probability $p_2 = 1 - p_1$. Let the stationary measure m be given by* (3.4.14)*. There is a value of p_1 so that IFS$(\{f_1, f_2\})$ has zero Lyapunov exponents with respect to m. For this value, intermittency near the invariant set S occurs.*

Sketch of proof. A further calculation on (3.4.15), using $\rho_2 = \rho_1/(\rho_1 - 1)$ and $p_2 = 1 - p_1$, shows $L = \ln(\rho_1 - 2) - (1 - p_1)\ln(\rho_1 - 1)$. We get $L = 0$ for

$$p_1 = 1 - \frac{\ln(\rho_1 - 2)}{\ln(\rho_1 - 1)}.$$

Note that the fixed point in $(0,1)$ of $x \mapsto ax(1-x)$ is unstable for $a > 3$. At $a = 1 + \sqrt{5}$, $x \mapsto ax(1-x)$ possesses a superstable period two orbit. Under the conditions of the theorem, f_1 possesses a stable period two orbit $\{q, f_1(q)\}$

with $1/2 < q < f_1(q)$. Write $J = [q, f_1(q)]$. The set $f_2 J \cup J$ is the union of two disjoint intervals, disjoint also from the critical point at $1/2$, and is invariant for IFS $(\{f_1, f_2\})$.

Consider the involution $R : [0,1] \to [0,1]$, $Rx = 1 - x$. Identify x with Rx. On J we have an iterated function system generated by f_1 and Rf_2. Observe that f_1 is monotone decreasing and Rf_2 is monotone increasing on J. Compare Figure 3.5. Using the methods of [13, Theorem 5.1 and Theorem 5.2] proves intermittency.

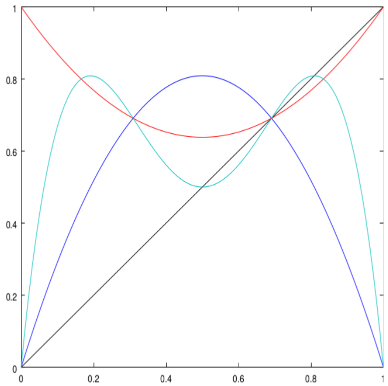

Figure 3.5: *The graphs of* $f_1(x) = rx(1-x)$ *with* $r = 1 + \sqrt{5}$, $f_1^2(x)$ *and* $Rf_2(x)$ *in one figure.*

Details are left to the reader. □

It would be interesting to consider the bifurcation scenario in which the probabilities and the logistic maps of Theorem 3.4.13 are varied.

Bibliography

[1] L. Arnold. *Random dynamical systems.* Springer-Verlag, 1998.

[2] K. B. Athreya, J. Dai. Random logistic maps I. *Journal of Theoretical Probability* 13, 595–608, 2000.

[3] K. B. Athreya, J. Dai. On the nonuniqueness of the invariant probability for i.i.d. random logistic maps. *The Annals of Probability* 30, 437–442, 2002.

[4] K. B. Athreya, H.J. Schuh. Random logistic maps II. The critical case. *Journal of Theoretical Probability* 16, 813–830, 2003.

[5] R.N. Bhattacharya, M. Majumdar. *Random dynamical systems.* Cambridge University Press, 2007.

[6] R.N. Bhattacharya, B.V. Rao, Random iterations of two quadratic maps. In: *Stochastics processes: a Festschrift in honour of Gopinath Kallianpur*, pp. 13–21. Springer Verlag, 1993.

[7] A. Bonifant, J. Milnor. Schwarzian derivatives and cylinder maps. in *Holomorphic dynamics and renormalization*, Fields Inst. Commun., 53, Amer. Math. Soc., pp. 1–21, 2008.

[8] N. Carlsson, A contractivity condition for iterated function systems. *Journal of Theoretical Probability*, 15, 613–630, 2002.

[9] H. Crauel. Extremal exponents of random dynamical systems do not vanish. *J. Dynam. Differential Equations* 2, 245–291, 1990.

[10] P. Diaconis, D. Freedman. Iterated random functions. *SIAM Review* 41, 4–76, 1999.

[11] M. Einsiedler, T. Ward. *Ergodic theory with a view towards number theory.* Springer Verlag, 2011.

[12] M. Gharaei, A.J. Homburg. Skew products of interval maps over subshifts. *J. Difference Equ. Appl.* 22, 941–958, 2016.

[13] M. Gharaei, A.J. Homburg. Random interval diffeomorphisms. *Discrete Cont. Dynam. Syst. Ser. S.* 10, 241–272, 2017.

[14] A. S. Gorodetskiĭ and Yu. S. Il′yashenko, Certain new robust properties of invariant sets and attractors of dynamical systems, *Funct. Anal. Appl.* 33, 95–105, 1999.

[15] J.F. Heagy, N. Platt, S.M. Hammel. Characterization of on-off intermittency. *Phys. Rev. E* 49, 1140–1150, 1994.

[16] A.J. Homburg. Synchronization in minimal iterated function systems on compact manifolds. preprint, 2016.

[17] J.E. Hutchinson. Fractals and self similarity. *Indiana Univ. Math. Journ.* 30, 713–747, 1981.

[18] Y. S. Il′yashenko. Thick attractors of step skew products. *Regular and Chaotic Dynamics* 15, 328–334, 2010.

[19] I. Kan. Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin. *Bull. Amer. Math. Soc.* 31:68–74, 1994.

[20] V. A. Kleptsyn, M. B. Nal′skiĭ. Contraction of orbits in random dynamical systems on the circle. *Funct. Anal. Appl.* 38:267–282, 2004.

[21] R. Mañé. *Ergodic theory and differentiable dynamics.* Springer-Verlag, 1987.

[22] W. de Melo, S. van Strien. *One-dimensional dynamics.* Springer Verlag, 1993.

[23] Ya. B. Pesin. *Lectures on partial hyperbolicity and stable ergodicity.* European Mathematical Society, 2004.

[24] G. Pianigiani. First return map and invariant measures. *Israel Journal of Mathematics* 35, 32–48, 1980.

[25] N. Platt, E.A. Spiegel, C. Tresser. On-off intermittency: a mechanism for bursting. *Phys. Rev. Letters* 70, 279–282, 1993.

[26] C. Pugh, M. Shub. Ergodic attractors. *Trans. Amer. Math. Soc.* 312, 1–54, 1989.

[27] A.N. Shiryayev. *Probability.* Springer Verlag, 1984.

[28] D. Steinsaltz. Random logistic maps and Lyapunov exponents. *Indag. Mathem.* 12, 557–584, 2001.

Chapter 4

Skew products of interval maps over subshifts

This chapter is:
M. Gharaei, A.J. Homburg. Skew products of interval maps over subshifts. *J. Difference Equ. Appl.* 22, 941-958, 2016.

Abstract

We treat step skew products over transitive subshifts of finite type with interval fibers. The fiber maps are diffeomorphisms on the interval; we assume that the end points of the interval are fixed under the fiber maps. Our paper thus extends work by V. Kleptsyn and D. Volk who treated step skew products where the fiber maps map the interval strictly inside itself.

We clarify the dynamics for an open and dense subset of such skew products. In particular we prove existence of a finite collection of disjoint attracting invariant graphs. These graphs are contained in disjoint areas in the phase space called trapping strips. Trapping strips are either disjoint from the end points of the interval (internal trapping strips) or they are bounded by an end point (border trapping strips). The attracting graphs in these different trapping strips have different properties.

4.1 Introduction

We aim to describe the dynamics of specific step skew products

$$(\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x))$$

with a shift as dynamics in the base and with interval fiber maps. That is, $\omega = (\omega_i)_{i\in\mathbb{Z}}$ is a sequence using finitely many symbols, and σ is the left shift operator acting on it. We treat such systems in cases where σ is a subshift of finite type and where the f_i's are diffeomorphisms on a compact interval that fix the endpoints of the interval.

Kleptsyn and Volk [5] conducted a study of dynamics of generic step skew products of diffeomorphisms on the line over subshifts of finite type. They looked at diffeomorphisms that are mapping a bounded interval strictly inside itself. They showed that so called bony graphs (after Kudryashov, see [6]) arise as attractors: these attractors are the union of a measurable graph and a zero measure set of intervals inside fibers (the bones).

A different situation occurs for diffeomorphisms on a compact interval that fix the endpoints of the interval. Such systems gained interest with an example by Kan [4] where they gave rise to intermingled basins. This example is over a full shift on two symbols and the end points of the interval are attracting on average. Il'yashenko [2, 3] similarly considered examples of diffeomorphisms over a full shift under an assumption of repulsion on average at the end points. He established attractors with positive standard measure (the standard measure is the product of Markov measure on the shift space and Lebesgue measure on the fiber space). The attractors are the closure of an invariant measurable graph. Note the contrast with bony graphs which have zero standard measure.

We provide a classification of dynamics of generic step skew products of diffeomorphisms on a compact interval (all diffeomorphisms fixing end points of the interval) over subshifts of finite type. Both types of graphs, bony and thick, can arise in a single step skew product.

4.1.1 Step skew product systems over subshifts of finite type

Write Ω for the finite set of symbols $\{1, \ldots, N\}$. Let $\mathcal{A} = (a_{ij})_{i,j=1}^{N}$ be a matrix with $a_{ij} \in \{0, 1\}$. Associated to $\mathcal{A}$ is the set $\Sigma_{\mathcal{A}}$ of bilateral sequences $\omega = (\omega_n)_{-\infty}^{\infty}$ composed of symbols in Ω and with transition matrix $\mathcal{A}$:

$$a_{\omega_n \omega_{n+1}} = 1$$

for all $n \in \mathbb{Z}$. Let $(\Sigma_{\mathcal{A}}, \sigma)$ be the subshift of finite type on $\Sigma_{\mathcal{A}}$. The map σ shifts every sequence $\omega \in \Sigma_{\mathcal{A}}$ one step to the left, $(\sigma\omega)_i = \omega_{i+1}$. We can also consider the left shift operator σ acting on the one-sided symbol space $\Sigma_{\mathcal{A}}^{+}$, i.e. the space of sequences $\omega = (\omega_n)_0^{\infty}$ composed of symbols in Ω with $a_{\omega_n \omega_{n+1}} = 1$ for all $n \geq 0$. The spaces $\Sigma_{\mathcal{A}}$ and $\Sigma_{\mathcal{A}}^{+}$ are endowed with the product topology. We assume that $\mathcal{A}$ is primitive, i.e.

$$\exists n_0 \in \mathbb{N} \; \forall i, j \in \Omega \; (\mathcal{A}^{n_0})_{ij} > 0.$$

This implies that the subshift σ is topologically transitive and topologically mixing.

Consider the interval $I = [0, 1]$ and $\{f_1, ..., f_N\}$, a finite family of orientation preserving (strictly increasing) C^2-diffeomorphisms defined on I assuming that $f_i(0) = 0$ and $f_i(1) = 1$ for every $i \in \Omega$. Write F^+ for the skew product system

$$F^+(\omega, x) = (\sigma\omega, f_\omega(x))$$

on $\Sigma_{\mathcal{A}}^+ \times I$, where the fiber maps f_ω depend only on ω_0, i.e. $f_\omega = f_{\omega_0}$. We also write $(F^+)^n(\omega, x) = (\sigma^n\omega, f_\omega^n(x))$ for iterates of F^+ in which

$$f_\omega^n(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x).$$

Likewise, on $\Sigma_{\mathcal{A}} \times I$ we have

$$F(\omega, x) = (\sigma\omega, f_\omega(x)).$$

In this paper we consider the following set of step skew product systems.

Definition 4.1.1. *We denote by* $\mathbf{S}$ *the set of step skew product systems* $F : \Sigma_{\mathcal{A}} \times I \to \Sigma_{\mathcal{A}} \times I$ *of the form*

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x)),$$

for orientation preserving diffeomorphisms $f_i : I \to I$ *that fix end points of* I*.*

4.1.2 Markov measures

Let $\Pi = (\pi_{ij})_{i,j=1}^N$ be a right stochastic matrix, i.e. $\pi_{ij} \geq 0$ and $\sum_{j=1}^N \pi_{ij} = 1$, such that $\pi_{ij} = 0$ precisely if $a_{ij} = 0$. By the Perron-Frobenius theorem for stochastic matrices, there exists a unique positive left eigenvector $p = (p_1, ..., p_N)$ for Π that corresponds to the eigenvalue 1; i.e.

$$\sum_{i=1}^N p_i \pi_{ij} = p_j, \ \forall j \in \Omega. \tag{4.1.1}$$

We assume that p is normalized so that it is a probability vector, $\sum_{i=1}^N p_i = 1$.

For a finite word $\omega_{k_1} ... \omega_{k_n}$, $k_i \in \mathbb{Z}$, the cylinder $C_{\omega_{k_1}, \ldots, \omega_{k_n}}^{k_1, \ldots, k_n}$ (we will also use the notation $C_\omega^{k_1, \ldots, k_n}$) is the set

$$C_{\omega_{k_1}, \ldots, \omega_{k_n}}^{k_1, \ldots, k_n} = \{\omega' \in \Sigma_{\mathcal{A}} \ ; \ \omega'_{k_i} = \omega_{k_i}, \ \forall 1 \leq i \leq n\}.$$

As cylinders form a countable base of the topology on $\Sigma_{\mathcal{A}}$, Borel measures on $\Sigma_{\mathcal{A}}$ are determined by their values on the cylinders. A Borel measure ν on $\Sigma_{\mathcal{A}}$ is called a Markov measure constructed from the distribution p_i and the transition probabilities π_{ij}, if for every $\omega \in \Sigma_{\mathcal{A}}$ and $k \leq l$,

$$\nu(C_\omega^{k, \ldots, l}) = p_{\omega_k} \prod_{i=k}^{l-1} \pi_{\omega_i \omega_{i+1}}.$$

One can easily check that with this definition ν is well-defined and is a probability measure. Moreover, ν is invariant under the shift map σ; it is ergodic and $\mathrm{supp}(\nu) = \Sigma_{\mathcal{A}}$. From now on, we consider a fixed ergodic Markov measure ν on $\Sigma_{\mathcal{A}}$. Write π for the natural projection $\Sigma_{\mathcal{A}} \mapsto \Sigma_{\mathcal{A}}^+$. Then, $\nu^+ = \pi\nu$ is the Markov measure on $\Sigma_{\mathcal{A}}^+$.

We do not consider measures on $\Sigma_{\mathcal{A}}$ that are not Markov measures. The reason is the connection of Markov measures to stationary measures for the stochastic process induced by F^+, see Section 4.3.

Definition 4.1.2. *The standard measure s on $\Sigma_{\mathcal{A}} \times I$ is the product of ν and the Lebesgue measure on the fiber.*

4.1.3 Trapping strips, bony graphs and thick graphs

Let $F \in \mathbf{S}$. As in [5], F admits forward invariant regions called trapping strips. Let $\varphi_1, \varphi_2 : \Sigma_{\mathcal{A}} \to I$, be continuous functions such that $\varphi_1 < \varphi_2$, i.e. $\varphi_1(\omega) < \varphi_2(\omega)$ for any $\omega \in \Sigma_{\mathcal{A}}$. Given such functions, we define the strip

$$\mathcal{S}_{\varphi_1,\varphi_2} = \{(\omega, x) \ ; \ \varphi_1(\omega) \leq x \leq \varphi_2(\omega)\}.$$

We distinguish two types of strips:

1. An internal strip has $0 < \varphi_1 < \varphi_2 < 1$;

2. For a border strip, $\varphi_1 = 0$ or $\varphi_2 = 1$, or both.

If the graph of φ_1 (or φ_2) is disjoint from $\Sigma_{\mathcal{A}} \times \{0\}$ (from $\Sigma_{\mathcal{A}} \times \{1\}$), then this graph is called an internal boundary.

Definition 4.1.3. *A strip $\mathcal{S}_{\varphi_1,\varphi_2}$ is called a trapping strip if $F(\mathcal{S}_{\varphi_1,\varphi_2}) \subseteq \mathcal{S}_{\varphi_1,\varphi_2}$. The strip $\mathcal{S}_{\varphi_1,\varphi_2}$ is called a strict trapping strip if moreover internal boundaries are mapped inside the interior of $\mathcal{S}_{\varphi_1,\varphi_2}$.*

Likewise one can consider trapping strips for F^+. It is clear that internal and border trapping strips are the only two possible kinds of trapping strips. Consider a trapping strip $\mathcal{S}$ with boundary functions $\varphi_1 < \varphi_2$. Because of monotonicity of the fiber maps, the images $F^n(\mathcal{S})$ are strips. Since for a trapping strip $\mathcal{S}$ also $F^n(\mathcal{S}) \subseteq \mathcal{S}$, we get that for every $n \geq 0$ the image $F^n(\mathcal{S})$ is a trapping strip. Therefore any trapping strip $\mathcal{S}$ has a non-empty maximal attractor

$$A_{\max} = \bigcap_{n=0}^{\infty} F^n(\mathcal{S}).$$

We encounter two different types of maximal attractors.

Definition 4.1.4. *A measurable graph B in $\Sigma_{\mathcal{A}} \times I$ is called a bony graph if it is contained in a closed set that intersects ν-almost every fiber in a single point and every other fiber in an interval, which is called a bone.*

Note that the standard measure of the closure of a bony graph is zero;

$$s\left(\overline{B}\right) = 0.$$

Following [5] we also call the closed set that is the union of the measurable graph and the bones, a bony graph. A bony graph can have an empty set of bones; a bony graph with an empty set of bones is a continuous graph. It is easy to construct examples where the maximal attractor is in fact a continuous graph.

Definition 4.1.5. *A measurable graph B in $\Sigma_{\mathcal{A}} \times I$ is called a thick graph if its closure has positive standard measure, i.e.*

$$s\left(\overline{B}\right) > 0.$$

We also call the closure of the thick graph, a thick graph.

4.2 Classification of dynamics for generic skew products

The Lyapunov exponent of a system $F \in \mathbf{S}$ at a point $(\omega, x) \in \Sigma_{\mathcal{A}} \times I$ is

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(f'_{\omega_{n-1}} \circ \cdots \circ f'_{\omega_0}(x)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln\left(f'_{\sigma^i\omega}(f^i_\omega(x))\right), \tag{4.2.1}$$

in case the limit exists. Since for every $i \in \Omega$, $x = 0, 1$ are fixed points of f_i, by the definition of Markov measure and Birkhoff's ergodic theorem, we obtain for $x = 0, 1$ that

$$L(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln\left(f'_{\sigma^i\omega}(x)\right) = \int_{\Sigma^+_{\mathcal{A}}} \ln\left(f'_\omega(x)\right)\, d\nu^+(\omega) = \sum_{i=1}^{N} p_i \ln\left(f'_i(x)\right)$$

for ν^+-almost all $\omega \in \Sigma^+_{\mathcal{A}}$. Note that generically $L(0)$ and $L(1)$ differ from zero.

We have introduced all notions needed to present our description of the dynamics of generic step skew product systems. The following theorem holds for step skew product systems from an open and dense subset of $\mathbf{S}$ which is given explicitly in Section 4.2.1 below.

Theorem 4.2.1. *There is an open and dense set $\mathbf{G}$ of $\mathbf{S}$, so that $F \in \mathbf{G}$ satisfies the following.*

F admits a finite collection of disjoint trapping strips $\mathcal{S}^t$, $1 \le t \le T$, of the form

$$\mathcal{S}^t = \cup_{k=1}^{N} C^0_k \times [A^t_k, B^t_k].$$

Furthermore,

1. *$\mathcal{S}^t$ contains a unique attracting invariant graph Γ^t: Γ^t is the graph of a measurable function $X^t : D^t \subset \Sigma_{\mathcal{A}} \to I$ defined on a set D^t with $\nu(D^t) = 1$. Given $x_i \in [A^t_i, B^t_i]$, for $\sigma^{-n}\omega \in D^t$,*

$$|f^n_{\sigma^{-n}\omega}(x_{\omega_{-n}}) - X^t(\omega)| \to 0 \text{ as } n \to \infty.$$

2. *If $L(0) < 0$, then $\Gamma = \Sigma_{\mathcal{A}} \times \{0\}$ is an attracting invariant graph: there is a set $D^t \subset \Sigma_{\mathcal{A}}$ with $\nu(D^t) = 1$, and a positive function $r : D^t \to (0, 1]$ so that for (ω, x) with $\omega \in D^t$, $0 \le x < r(\omega)$,*

$$f^n_\omega(x) \to 0 \text{ as } n \to \infty.$$

   A similar statement applies to $\Sigma_{\mathcal{A}} \times \{1\}$ if $L(1) < 0$.

3. (a) *if the strict trapping strip is a border trapping strip, then its maximal attractor is a thick graph.*
   (b) *if the strict trapping strip is an internal trapping strip, then its maximal attractor is a bony graph.*

4. *With respect to the standard measure on $\Sigma_{\mathcal{A}} \times I$, the positive orbit of almost every initial point converges to one of the attracting graphs from items (1), (2).*

Kleptsyn and Volk [5] show that the bony graphs in internal strict trapping strips are upper-semicontinuous:

$$\forall \omega \in \Sigma_{\mathcal{A}}\, \forall \varepsilon > 0\, \exists \delta > 0\, d(\omega, \omega') < \delta \Rightarrow B_{\omega'} \subset U_\varepsilon(B_\omega),$$

where d metrizes the product topology on $\Sigma_{\mathcal{A}}$, $B_\omega = B \cap (\{\omega\} \times I)$ and $U_\varepsilon(B_\omega)$ denotes the ε-neighborhood of B_ω in $\{\omega\} \times I$. They refer to these bony graphs as continuous bony graphs.

4.2.1 Genericity conditions

The open and dense set $\mathbf{G}$ of $\mathbf{S}$ in Theorem 4.2.1 is determined by a number of genericity conditions. Here we list the imposed genericity conditions. They are equal to those appearing in [5], with two additional conditions related to the fixed boundary points of I (items (1) and (5) below). The first condition gives that the end points of I are repelling or attracting, on average.

1. $L(0), L(1) \neq 0$.

To formulate the further conditions we introduce the notions of simple transition and simple return.

Definition 4.2.2. *A finite word $\omega_1 \dots \omega_n$ is called admissible if each pair of consecutive symbols $\omega_i \omega_{i+1}$ is admissible; i.e. $\pi_{\omega_i \omega_{i+1}} \neq 0$. A map of the form*

$$f_{\omega_1 \dots \omega_n} := f_{\omega_n} \circ \dots \circ f_{\omega_1} : I \to I,$$

is called an admissible composition if the word $\omega_1 \dots \omega_n$ is admissible.

Definition 4.2.3. *A simple transition is an admissible composition $f_{\omega_1 \dots \omega_n} : I \to I$ in which all the symbols ω_i, $1 \leq i \leq n$ are different. It is called a simple return if also $\omega_1 = \omega_{n+1}$.*

We can now state the following genericity conditions.

2. Any fixed point q of any simple return g is hyperbolic: $g'(q) \neq 1$;

and if we consider the restriction of f_i's to the open interval $(0, 1)$ then

3. No attracting fixed point of a simple return is mapped to a repelling fixed point of a simple return by a simple transition. Also, no repelling fixed point of a simple return is mapped to an attracting fixed point of a simple return by a simple transition;

4. One can not choose from the interior of each interval I_k, $k \in \Omega$, a single point a_k such that for any admissible couple i, j one could have $f_i(a_i) = a_j$.

Condition (4) precludes finite invariant sets, see [5]. The final condition relates to minimal iterated function systems. First we recall the definition of minimality of an iterated function system. Suppose given an iterated function system IFS $\{g_1, \ldots, g_k\}$ of continuous maps g_i on a metric space X. Let Y be a subset of X with $g_i(Y) \subset Y$ for all i. We say that IFS $\{g_1, \ldots, g_k\}$ is minimal on Y if for every points $x, y \in Y$ and every neighborhood V of y, there is a composition $g_{i_n} \circ \cdots \circ g_{i_1}$ that maps x into V.

The proof of [2, Lemma 3] gives the following result.

Proposition 4.2.4. *Let $f, g : I \to I$ be diffeomorphisms fixing the boundary points of I. Assume that $\lambda = f'(0) < 1$, $\mu = g'(0) > 1$. Assume further that either*

$$\ln(\lambda)/\ln(\mu) \notin \mathbb{Q},$$

or

$$\frac{f''(0)}{\lambda^2 - \lambda} \neq \frac{g''(0)}{\mu^2 - \mu}.$$

Then the iterated function system generated by f, g is minimal on some interval $(0, u)$.

Proof. Il'yashenko [2, Lemma 3] considers, for $x, y \in (0,1)$, compositions $g^l \circ f^k(x)$ that converge to y for suitable $k, l \to \infty$. His analysis uses linearizing coordinates $h \circ f \circ h^{-1}(x) = \lambda x$ with $x \in [0, s]$ for an $s < 1$. Here h is a local diffeomorphism. The two cases where $\ln(\lambda), \ln(\mu)$ are rationally dependent or not, are distinguished. In case $\ln(\lambda), \ln(\mu)$ are rationally dependent, the argument works if the second order derivative of $h \circ g \circ h^{-1}$ at 0 is not zero. An explicit calculation shows that this gives the condition in the proposition. □

5. The admissible returns f, g introduced in Lemma 4.4.12 satisfy the conditions formulated in Proposition 4.2.4.

4.3 Stationary measures

A key role in our study is played by ergodic invariant measures for the skew product systems. The necessary material is collected in this section.

Write $\mathcal{I} = \Omega \times I$. For every $i, j \in \Omega$, π_{ij} equals the probability of the transition from a point (i, x) in $\mathcal{I}$ to another point $(j, f_i(x))$. For every $i \in \Omega$ we denote $\{i\} \times I \in \mathcal{I}$ by I_i. We can identify I_i with I. Denote by $\mathcal{B}$ the Borel sigma-algebra on I. We consider Borel probability measures $\mathfrak{m}$ on the space $\mathcal{I}$ with $\mathfrak{m}(I_i) = p_i$. For such a measure $\mathfrak{m}$, define the probability measure $\mathfrak{m}_i$ on I_i by

$$\mathfrak{m}_i = \frac{\mathfrak{m}|_{I_i}}{\mathfrak{m}(I_i)}.$$

We denote by $f_i\mathfrak{m}_i$ the push-forward measure of $\mathfrak{m}_i$ by f_i, where $f_i\mathfrak{m}_i(B) = \mathfrak{m}_i(f_i^{-1}(B))$ for $\mathcal{B}$-measurable sets B. Define $\mathcal{T}$ on the space of probability measures on $\mathcal{I}$ by

$$(\mathcal{T}\mathfrak{m})_k = \frac{1}{p_k}\sum_{i=1}^{N} p_i \pi_{ik} f_i \mathfrak{m}_i, \ \forall k \in \Omega,$$

with an understanding that $\mathcal{T}\mathfrak{m}(I_i) = p_i$.

Definition 4.3.1. *A measure $\mathfrak{m}$ on the space $\mathcal{I}$ is stationary if $\mathcal{T}\mathfrak{m} = \mathfrak{m}$.*

Recall the notation $C_k^0 = \{\omega \in \Sigma_{\mathcal{A}} \mid \omega_0 = k\}$. Write $C_k^{+,0} = \{\omega \in \Sigma_{\mathcal{A}}^+ \mid \omega_0 = k\}$. For $k \in \Omega$, write ν_k^+ for the restriction of the Markov measure ν^+ to the cylinder $C_k^{+,0}$. A direct computation gives the following correspondence between stationary measures and invariant measures for the skew product system with one sided time.

Lemma 4.3.2. *A probability measure $\mathfrak{m}$ is a stationary probability measure if and only if μ^+ defined by*

$$\mu^+ = \sum_{k=1}^{N} \nu_k^+ \times \mathfrak{m}_k \tag{4.3.1}$$

is an invariant measure of F^+ with marginal ν^+ on $\Sigma_{\mathcal{A}}^+$.

Let $\mathcal{F}^+$ be the Borel sigma-algebra on $\Sigma_{\mathcal{A}}^+$. It yields a sigma-algebra $\mathcal{F}_0 = \pi^{-1}\mathcal{F}^+$ on $\Sigma_{\mathcal{A}}$, where $\pi : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}^+$ is the natural coordinate projection. Write $\mathcal{F}$ for the Borel sigma-algebra on $\Sigma_{\mathcal{A}}$. A measure μ on $\Sigma_{\mathcal{A}} \times I$ with marginal ν has conditional measures μ_ω on the fibers $\{\omega\} \times I$, such that

$$\mu(A) = \int_{\Sigma_{\mathcal{A}}} \mu_\omega(A \cap (\{\omega\} \times I))\, d\nu(\omega)$$

for measurable sets A. A measure μ^+ on $\Sigma_{\mathcal{A}}^+ \times I$ with marginal ν^+ likewise has conditional measures μ_ω^+. It is convenient to consider ν^+ also as a measure on $\Sigma_{\mathcal{A}}$ with sigma-algebra $\mathcal{F}_0$ and μ^+ also as a measure on $\Sigma_{\mathcal{A}} \times I$ with sigma-algebra $\mathcal{F}_0 \otimes \mathcal{B}$. When $\omega \in \Sigma_{\mathcal{A}}$ we will write μ_ω^+ for the conditional measures $\mu_{\pi\omega}^+$. The spaces of measures are equipped with the weak star topology. The following result relates invariant measures for the one-sided and the two-sided skew product systems. It is a special case of [1, Theorem 1.7.2]. We write $\Sigma_{\mathcal{A}} = \Sigma_{\mathcal{A}}^- \times \Sigma_{\mathcal{A}}^+$. and with this $\omega = (\omega^-, \omega^+)$ for $\omega \in \Sigma_{\mathcal{A}}$.

Proposition 4.3.3. *Let μ^+ be an F^+-invariant probability measure with marginal ν^+. Then there exists an F-invariant probability measure μ with marginal ν and conditional measures*

$$\mu_\omega = \lim_{n\to\infty} f^n_{\sigma^{-n}\omega} \mu^+_{\sigma^{-n}\omega}, \tag{4.3.2}$$

ν-almost surely.

Let μ be an F-invariant probability measure with marginal ν and $\Pi : \Sigma_{\mathcal{A}}^- \times \Sigma_{\mathcal{A}}^+ \times I \to \Sigma_{\mathcal{A}}^+ \times I$ be the natural projection where $\Sigma_{\mathcal{A}} = \Sigma_{\mathcal{A}}^- \times \Sigma_{\mathcal{A}}^+$. Then

$$\mu^+ = \Pi\mu \tag{4.3.3}$$

is an F^+-invariant probability measure with marginal ν^+.

The correspondence $\mu \leftrightarrow \mu^+$ given by (4.3.2), (4.3.3) *is one-to-one and μ is ergodic if and only if μ^+ is ergodic. An invariant measure μ for which μ_ω depends on the past $\omega^- \in \Sigma_{\mathcal{A}}^{-1}$ only, corresponds to a measure μ^+ that comes from a stationary measure $\mathfrak{m}$ as in* (4.3.1).

4.4 Bony graphs and thick graphs

The proof of Theorem 4.2.1 is divided into different steps. We will first discuss the case where both $L(0) > 0$ and $L(1) > 0$. The other cases are then easy to treat and will be considered later.

4.4.1 Repelling end points

We assume $L(0) > 0$ and $L(1) > 0$. We briefly outline the different steps in the proof of Theorem 4.2.1, which will be worked out below.

Step 1: Stationary measures By a Krylov-Bogolyubov procedure on a suitable class of probability measures we construct stationary measures that do not assign measure to the endpoints 0 or 1 of the interval $[0, 1]$.

Step 2: Trapping strips The convex hull of the support of an ergodic stationary measure, as constructed in the first step, provides a trapping strip. Trapping strips can be border trapping strips or internal trapping strips.

Step 3: Conditional measures A stationary measure gives rise to an invariant measure of the skew product system with two sided time. We prove that such an invariant measure has delta measures as conditional measures on fibers. For each trapping strip there is a unique invariant measure with support in the trapping strip.

Step 4: Attracting graphs The points of the delta measures constitute an invariant graph. We discuss its properties in this final step.

For internal trapping strips these results have been obtained by Kleptsyn and Volk [5]. We now elaborate the different steps.

Step 1: Stationary measures.

In the construction of stationary measures we iterate the transformation $\mathcal{T}$, whose fixed points are the stationary measures. For $k \in \Omega$ and for any $n \in \mathbb{N}$, the nth iterate of $\mathfrak{m}$ under the transformation $\mathcal{T}$ is calculated on I_k as

$$(\mathcal{T}^n \mathfrak{m})_k = \frac{1}{p_k} \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \pi_{i_n k} f^n_{i_1 \ldots i_n} \mathfrak{m}_{i_1}. \tag{4.4.1}$$

The above sum is over all N^n possible symbol sequences of length $n+1$ in Ω^n ending with the symbol k, and $p_{i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \pi_{i_n k}$ is the probability of the transition to the symbol k in n steps along the symbol sequence $i_1, \ldots, i_n, k$.

We will need the following arithmetic bound that is connected to formula (4.4.1). Recall the assumptions $L(0) > 0$ and $L(1) > 0$. Write $\lambda_i = f_i'(0)$ and $\bar{\lambda}_i = f_i'(1)$.

Lemma 4.4.1. *For n large enough and any k, $1 \leq k \leq N$,*

$$\frac{1}{p_k} \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \pi_{i_n k} \frac{1}{n} \ln(\lambda_{i_1} \cdots \lambda_{i_n}) > 0, \tag{4.4.2}$$

$$\frac{1}{p_k} \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \pi_{i_n k} \frac{1}{n} \ln(\bar{\lambda}_{i_1} \cdots \bar{\lambda}_{i_n}) > 0. \tag{4.4.3}$$

Proof. We consider the end point 0. First note that for ν^+-almost all ω,

$$\begin{aligned}
L(0) &= \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \ln(f'_{\sigma^i \omega}(0)) \\
&= \lim_{k\to\infty} \frac{1}{kn} \sum_{i=0}^{k-1} \ln((f^n_{\sigma^{ni}\omega})'(0)) \\
&= \frac{1}{n} \int_{\Sigma^+_{\mathcal{A}}} \ln((f^n_\omega)'(0))\, d\nu^+(\omega) \\
&= \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \frac{1}{n} \ln(f'_{i_1}(0) \cdots f'_{i_n}(0)).
\end{aligned}$$

Hence

$$L(0) = \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1} \pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n} \frac{1}{n} \ln(\lambda_{i_1} \cdots \lambda_{i_n}). \tag{4.4.4}$$

A similar equality as (4.4.4) holds for $L(1)$, the Lyapunov exponent at $x = 1$.

The sum in (4.4.2) is an average over all symbol sequences of length $n+1$ ending with a symbol k:

$$\begin{aligned}\frac{1}{p_k} \sum_{i_1,\ldots,i_n=1}^{N} & p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\pi_{i_n k}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n}) \\ &= \frac{1}{p_k}\int_{P_{n,k}} \frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})\, d\nu^+(i),\end{aligned}$$

where $i=(i_1,\ldots)$ and $P_{n,k}=\{i\in\Sigma_{\mathcal{A}}^+ \ ; \ \sigma^{n+1}i\in C_k\}$. Since ν^+ is invariant we have $\nu^+(C_k)=\nu^+(\sigma^{-(n+1)}(C_k))=\nu^+(P_{n,k})$ for any $n\in\mathbb{N}$. We observe that $\nu^+(P_{n,k})=p_k$ independent of n and we suppress the dependence of $P_{n,k}$ to n.

Write

$$\Gamma(\varepsilon,M)=\left\{\omega\in\Sigma_{\mathcal{A}}^+ \ ; \ \left|\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})-L(0)\right|<\varepsilon \text{ for } n\geq M\right\}.$$

By ergodicity (4.2.1), $\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})$ converges to $L(0)$ for ν^+-almost all $(i_1,\ldots)\in\Sigma_{\mathcal{A}}^+$, as $n\to\infty$. We therefore have that for all $\varepsilon>0$ there exists M so that $\nu^+(\Gamma(\varepsilon,M))>1-\varepsilon$.

Take the positive constant K such that $|\ln\lambda_j-L(0)|\leq K$ for all j. Choose ε small and $M=M(\varepsilon)$ so that $\nu^+(\Gamma(\varepsilon,M))>1-\varepsilon$. Write $\Gamma(\varepsilon,M)^c=\Sigma_{\mathcal{A}}^+\setminus\Gamma(\varepsilon,M)$. For any $n\geq M$ we can compute,

$$\begin{aligned} d_n &= \left|\frac{1}{p_k}\int\limits_{P_k}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})\,d\nu^+(i)-L(0)\right| \\ &\leq \frac{1}{p_k}\int\limits_{P_k\cap\Gamma(\varepsilon,M)}\left|\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})-L(0)\right|d\nu^+(i) \\ &\quad+\frac{1}{p_k}\int\limits_{P_k\cap\Gamma(\varepsilon,M)^c}\left|\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})-L(0)\right|d\nu^+(i) \\ &\leq \varepsilon+K\varepsilon.\end{aligned}$$

Therefore, $d_n\to 0$, as $n\to\infty$. Likewise,

$$\left|\frac{1}{p_k}\int\limits_{P_k}\frac{1}{n}\ln(\bar\lambda_{i_1}\cdots\bar\lambda_{i_n})\,d\nu^+(i)-L(1)\right|\to 0, \text{ as } n\to\infty.$$

Since $L(0)$ and $L(1)$ are positive, for n large both

$$\frac{1}{p_k}\int_{P_k}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})\,d\nu^+(i)>0$$

and

$$\frac{1}{p_k}\int_{P_k}\frac{1}{n}\ln(\bar\lambda_{i_1}\cdots\bar\lambda_{i_n})\,d\nu^+(i)>0.$$

□

Let $\mathcal{M}$ be the space of all Borel probability measures on $\mathcal{I}$ endowed with the weak-star topology. For small $0 < \alpha < 1$, $q > 0$ and $c > 0$ define

$$\mathcal{N}_c = \{\mathfrak{m} \in \mathcal{M};\ \forall\, 0 \le x \le q,\ \mathfrak{m}_k\big([0,x)\big) \le cx^\alpha \text{ and } \mathfrak{m}_k\big((1-x,1]\big) \le cx^\alpha\ \forall k \in \Omega\}.$$

The condition on the measure of small intervals $[0,x)$ and $(1-x,1]$ excludes measures supported on the end points 0 and 1. Note that $\mathcal{N}_c$ depends on α and q; but we do not include this dependence in the notation. We first show that there exist ergodic stationary measures which belong to $\mathcal{N}_c$.

Proposition 4.4.2. *Under the assumptions of Theorem 4.2.1 and in particular $\sum_{i=1}^N p_i \ln f_i'(x) > 0$ for $x = 0, 1$, there exist positive α, c, q and $n_1 \in \mathbb{N}$ such that $\mathcal{T}^{n_1}\mathcal{N}_c \subset \mathcal{N}_c$.*

Proof. Note that by (4.1.1), for each $k \in \Omega$,

$$\sum_{i_1,\dots,i_n=1}^{N} p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\pi_{i_n k} = p_k. \tag{4.4.5}$$

Let n_1 be a number such that for any $n \ge n_1$ the inequality (4.4.2) holds in Lemma 4.4.1. In the following, fix any $n \ge n_1$. Since for each k there are N^n possible transitions in $n+1$ steps ending with k we may rewrite (4.4.2) as

$$\sum_{i=1}^{N^n} \rho_i^k \ln\gamma_i > 0$$

in which $\sum_{i=1}^{N^n} \rho_i^k = 1$ by (4.4.5). We claim that there is a small $\alpha > 0$ such that our assumption $\sum_{i=1}^{N^n} \rho_i^k \ln\gamma_i > 0$ implies $\sum_{i=1}^{N^n} \rho_i^k \gamma_i^{-\alpha} < 1$. Namely, since $\lim\limits_{\alpha\to 0} \frac{1-\gamma_i^{-\alpha}}{\alpha} = \ln\gamma_i$, $1 \le i \le N^n$, $\sum_{i=1}^{N^n} \rho_i^k \ln\gamma_i > 0$ implies that for sufficiently small $\alpha > 0$,

$$\sum_{i=1}^{N^n} \rho_i^k \frac{1-\gamma_i^{-\alpha}}{\alpha} > 0.$$

Multiplying by α we get

$$\sum_{i=1}^{N^n} \rho_i^k - \sum_{i=1}^{N^n} \rho_i^k \gamma_i^{-\alpha} > 0,$$

which implies $\sum_{i=1}^{N^n} \rho_i^k \gamma_i^{-\alpha} < 1$, because $\sum_{i=1}^{N^n} \rho_i^k = 1$.

A similar reasoning applies to the end point 1 of I, starting with (4.4.3) rewritten as $\sum_{i=1}^{N^n} \rho_i^k \ln\bar{\gamma}_i > 0$, to show that for α small, also $\sum_{i=1}^{N^n} \rho_i^k \bar{\gamma}_i^{-\alpha} < 1$.

Thus, there exists a small $\delta > 0$ so that

$$\sum_{i=1}^{N^n} \frac{\rho_i^k}{(\gamma_i - \delta)^\alpha} < 1 \tag{4.4.6}$$

and likewise $\sum_{i=1}^{N^n} \frac{\rho_i^k}{(\bar{\gamma}_i-\delta)^\alpha} < 1$. Moreover, for such $\delta > 0$ we are able to choose a sufficiently small $q = q(\delta) > 0$ in such a way that for each symbol sequence $i_1, \ldots, i_n$ in Ω^n,

$$f_{i_1,\ldots,i_n}^{-n}(x) \leq \frac{x}{(\lambda_{i_1}\ldots\lambda_{i_n}) - \delta}, \; \forall\, 0 \leq x \leq q. \tag{4.4.7}$$

Take c with $cq^\alpha > 1$. Take a measure $\mathfrak{m}$ from the $\mathcal{N}_c$ that corresponds to α and q. We will prove $\mathcal{T}^n\mathfrak{m} \in \mathcal{N}_c$. To do this we must show that if $x \leq q$ then $(\mathcal{T}^n\mathfrak{m})_k\big([0,x)\big) \leq cx^\alpha$ and $(\mathcal{T}^n\mathfrak{m})_k\big((1-x,1]\big) \leq cx^\alpha$ for all $k \in \Omega$. Knowing that $\mathfrak{m}_k\big([0,x)\big) \leq cx^\alpha$ for each $k \in \Omega$ and applying (4.4.6), (4.4.7) we get:

$$\begin{aligned}
(\mathcal{T}^n\mathfrak{m})_k\big([0,x)\big) &= \sum_{i_1,\ldots,i_n=1}^{N} \frac{1}{p_k} p_{i_1}\pi_{i_1i_2}\ldots\pi_{i_{n-1}i_n}\pi_{i_ni_k} f^n_{i_1\ldots i_n}\mathfrak{m}_{i_1}\big([0,x)\big) \\
&= \sum_{i_1,\ldots,i_n=1}^{N} \frac{1}{p_k} p_{i_1}\pi_{i_1i_2}\ldots\pi_{i_{n-1}i_n}\pi_{i_ni_k}\mathfrak{m}_{i_1}\big(f^{-n}_{i_1\ldots i_n}[0,x)\big) \\
&\leq \sum_{i_1,\ldots,i_n=1}^{N} \frac{1}{p_k} p_{i_1}\pi_{i_1i_2}\ldots\pi_{i_{n-1}i_n}\pi_{i_ni_k}\mathfrak{m}_{i_1}\big([0, \frac{x}{(\lambda_{i_1}\ldots\lambda_{i_n}) - \delta})\big) \\
&\leq \sum_{i=1}^{N^n} \rho_i^k c\Big(\frac{x}{\gamma_i - \delta}\Big)^\alpha \\
&= c\left(\sum_{i=1}^{N^n} \frac{\rho_i^k}{(\gamma_i-\delta)^\alpha}\right)x^\alpha \\
&\leq cx^\alpha.
\end{aligned} \tag{4.4.8}$$

Likewise, $(\mathcal{T}^n\mathfrak{m})_k\big((1-x,1]\big) \leq cx^\alpha$ for $x \leq q$. Thus, for every $\mathfrak{m} \in \mathcal{N}_c$, the image $\mathcal{T}^n\mathfrak{m}$ belongs to $\mathcal{N}_c$. □

Now we know that $\mathcal{T}^{n_1}(\mathcal{N}_c) \subset \mathcal{N}_c$. By the Krylov-Bogolyubov averaging method, for a measure $\mathfrak{m} \in \mathcal{N}_c$ on the compact metric space $\mathcal{I}$ there is a subsequence of $\{\frac{1}{n}\sum_{r=0}^{n-1}\mathcal{T}^{rn_1}\mathfrak{m}\}_{n\in\mathbb{N}}$ which is convergent to a probability measure $\hat{\mathfrak{m}} \in \mathcal{N}_c$ such that $\mathcal{T}^{n_1}\hat{\mathfrak{m}} = \hat{\mathfrak{m}}$. Note that

$$\bar{\mathfrak{m}} = \frac{1}{n_1}\big(\hat{\mathfrak{m}} + \mathcal{T}\hat{\mathfrak{m}} + \ldots + \mathcal{T}^{n_1-1}\hat{\mathfrak{m}}\big)$$

is a probability measure. Since $\mathcal{T}$ is linear and $\mathcal{T}^{n_1}\hat{\mathfrak{m}} = \hat{\mathfrak{m}}$, the measure $\bar{\mathfrak{m}}$ is a fixed point of $\mathcal{T}$:

$$\mathcal{T}\bar{\mathfrak{m}} = \frac{1}{n_1}\left(\mathcal{T}\hat{\mathfrak{m}} + \mathcal{T}^2\hat{\mathfrak{m}} + \ldots + \mathcal{T}^{n_1}\hat{\mathfrak{m}}\right) = \bar{\mathfrak{m}}.$$

We have found a stationary measure $\bar{\mathfrak{m}}$ in $\mathcal{N}_c$ for some c.

The following additional reasoning shows that there is an ergodic stationary measure in $\mathcal{N}_c$. Let $\mathcal{N}$ be the set of stationary measures on $\mathcal{I}$ which is a convex compact subset of $\mathcal{M}$. The ergodic stationary measures are the extreme points of it. Note that $\mathcal{N}_c \cap \mathcal{N}$ is a convex compact subset of $\mathcal{N}$, which is itself also convex and compact. We claim that the extreme points of $\mathcal{N}_c \cap \mathcal{N}$ are extreme points of $\mathcal{N}$. Suppose by contradiction that there are $\bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2 \in \mathcal{N} \setminus (\mathcal{N}_c \cap \mathcal{N})$ and the convex combination $\bar{\mathfrak{m}} = s\bar{\mathfrak{m}}_1 + (1-s)\bar{\mathfrak{m}}_2 \in \mathcal{N}_c \cap \mathcal{N}$. In this case, for $0 \le x \le q$, $\bar{\mathfrak{m}}_{1,k}([0,x)) \le (c/s)x^\alpha$ and $\bar{\mathfrak{m}}_{1,k}((1-x,1]) \le (c/s)x^\alpha$ and similar estimates for $\bar{\mathfrak{m}}_2$. That is, $x \mapsto \bar{\mathfrak{m}}_{i,k}([0,x))/x^\alpha$ and $x \mapsto \bar{\mathfrak{m}}_{i,k}((1-x,1])/x^\alpha$ are bounded. As $\mathcal{T}\bar{\mathfrak{m}} = \bar{\mathfrak{m}}$, we have by (4.4.6), (4.4.8) that $\bar{\mathfrak{m}} \in \mathcal{N}_{\tilde{c}}$ for some $\tilde{c} < c$. It follows that $t\bar{\mathfrak{m}}_1 + (1-t)\bar{\mathfrak{m}}_2 \in \mathcal{N}_c$ for t close to s. So s is an interior point of the set of values t for which $t\bar{\mathfrak{m}}_1 + (1-t)\bar{\mathfrak{m}}_2 \in \mathcal{N}_c \cap \mathcal{N}$. Since $\mathcal{N}_c \cap \mathcal{N}$ is closed it follows that $\bar{\mathfrak{m}}_i \in \mathcal{N}_c \cap \mathcal{N}$ and the claim is proved. Since the extreme points of $\mathcal{N}$ are ergodic stationary measures, we conclude that the extreme points of $\mathcal{N}_c \cap \mathcal{N}$ are ergodic stationary measures. Since the set of extreme points of $\mathcal{N}_c \cap \mathcal{N}$ is nonempty by the Krein-Milman theorem, there are ergodic stationary measures.

Step 2: Trapping strips.

Recall from Lemma 4.3.2 that a stationary measure $\mathfrak{m}$ gives rise to an invariant measure for the one-sided skew product system, with marginal ν^+ on $\Sigma_{\mathcal{A}}^+$. We will see that the supports of such invariant measures are contained in mutually disjoint trapping strips. This step closely follows [5], with adjustments to account for the fixed end points.

Definition 4.4.3. *A subset $\mathcal{D} = \bigcup_{k=1}^N \mathcal{D}_k \subseteq \mathcal{I}$ is called a domain if for each $k \in \Omega$, $\mathcal{D}_k$ is a closed interval in I_k.*

A boundary point of an interval $\mathcal{D}_k$ different from 0 or 1 is called an internal boundary point.

Definition 4.4.4. *A domain $\mathcal{D} = \bigcup_{k=1}^N \mathcal{D}_k \subseteq \mathcal{I}$ is trapping if any admissible map takes it to itself,*

$$\forall k, l: \ \pi_{kl} > 0, \ f_k(\mathcal{D}_k) \subseteq \mathcal{D}_l.$$

The domain is strict trapping if any internal boundary point of $\mathcal{D}_k$ is mapped inside the interior of $\mathcal{D}_l$.

The following proposition is [5, Proposition 4.5] and holds also here.

Proposition 4.4.5. *The following conditions are equivalent:*

i) the domain $\mathcal{D} = \bigcup_{k=1}^N \mathcal{D}_k \subseteq \mathcal{I}$ is (strict) trapping;

ii) the strip $\mathcal{S}^+ = \bigcup_{k=1}^N C_k^{+,0} \times \mathcal{D}_k \subseteq \Sigma_{\mathcal{A}}^+ \times I$ is (strict) trapping for the skew product F^+;

iii) the strip $\mathcal{S} = \bigcup_{k=1}^N C_k^0 \times \mathcal{D}_k \subseteq \Sigma_{\mathcal{A}} \times I$ is (strict) trapping for F.

Consider an arbitrary ergodic stationary measure $\mathfrak{m} \in \mathcal{N}_c$. Denote the interval that spans the support of $\mathfrak{m}_k$ by $I_{\mathfrak{m},k} = [A_{\mathfrak{m},k}, B_{\mathfrak{m},k}]$:

$$\begin{aligned} A_{\mathfrak{m},k} &= \min \operatorname{supp}(\mathfrak{m}_k), \\ B_{\mathfrak{m},k} &= \max \operatorname{supp}(\mathfrak{m}_k). \end{aligned}$$

For every admissible i, j we have $f_i(\operatorname{supp}(\mathfrak{m}_i)) \subseteq \operatorname{supp}(\mathfrak{m}_j)$. Since the maps f_i are monotone we have that for any admissible transition i, j,

$$f_i(I_{\mathfrak{m},i}) \subseteq I_{\mathfrak{m},j}. \tag{4.4.9}$$

Therefore, the collection $\mathcal{I}_{\mathfrak{m}} = \bigcup_{k=1}^{N} I_{\mathfrak{m},k}$ is a domain, which is trapping by (4.4.9).

The imposed genericity conditions imply that for a trapping domain no interval $I_{\mathfrak{m},k}$ can be a single point.

Lemma 4.4.6. *Consider an arbitrary trapping domain $\mathcal{I}_{\mathfrak{m}}$. Then either $A_{\mathfrak{m},k} = 0$ for all $k \in \Omega$, or $A_{\mathfrak{m},k} \neq 0$ for all $k \in \Omega$. In the latter case, there exist an attracting fixed point A of a simple return and a simple transition f such that $A_{\mathfrak{m},k} = f(A)$. In the former case, i.e. if $A_{\mathfrak{m},k} = 0$ for all $k \in \Omega$, then 0 is an attracting fixed point of a simple return. An analogous statement holds for $B_{\mathfrak{m},k}$.*

Proof. For a chosen trapping domain $\mathcal{I}_{\mathfrak{m}}$ suppose that $A_{\mathfrak{m},k} = 0$ for some $k \in \Omega$. Then, knowing that $x = 0$ is a fixed point of f_k for all k, we have for any $l \in \Omega$ such that k, l is admissible that

$$0 = f_k(0) = f_k(\min \operatorname{supp}(\mathfrak{m}_k)) \in f_k(\operatorname{supp}(\mathfrak{m}_k)) \subseteq \operatorname{supp}(\mathfrak{m}_l).$$

Hence, $A_{\mathfrak{m},l} = \min \operatorname{supp}(\mathfrak{m}_l) = 0$. Since the subshift σ is transitive $A_{\mathfrak{m},k} = 0$ for all $k \in \Omega$.

If $A_{\mathfrak{m},k} \neq 0$ for all k, [5, Lemma 6.3] applies and the result for $A_{\mathfrak{m},k}$ holds by that lemma. If $A_{\mathfrak{m},k} = 0$ for all k, the arguments of [5, Lemma 6.3] apply to yield the same conclusion (the simple transition is redundant since 0 is a fixed point of all maps). □

By Birkhoff's ergodic theorem, a generic sequence of random iterations (k_n, x_n), $x_n \in I_{k_n}$ of a $\mathfrak{m}$-generic initial point is distributed with respect to the measure $\mathfrak{m}$. If we choose such a generic initial point (k_0, x_0) then because the points (k_n, x_n) are distributed with respect to $\mathfrak{m}$, the set $X_k = \{x_n\}|_{k_n=k}$ is dense in $\operatorname{supp}(\mathfrak{m}_k)$ for any k. We apply this observation in the proof of the next lemma, which corresponds to [5, Lemma 6.7].

Lemma 4.4.7. *For any two trapping domains $\mathcal{I}_{\mathfrak{m}_1}$ and $\mathcal{I}_{\mathfrak{m}_2}$ of two ergodic stationary measures $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{N}_c$ the corresponding intervals $I_{\mathfrak{m}_1,k}$ and $I_{\mathfrak{m}_2,k}$ are either disjoint for any k or coincide for any k.*

Proof. Assume that the intervals $I_{\mathfrak{m}_1,k}$ and $I_{\mathfrak{m}_2,k}$ intersect but do not coincide. Then, there is at least one end point of one of them that does not belong to the other one. Without loss of generality let it be the point $B_{\mathfrak{m}_1,k}$. There is a neighborhood V of $B_{\mathfrak{m}_1,k}$ such that $I_{\mathfrak{m}_2,k} \cap V = \emptyset$.

By genericity condition (4), $A_{\mathfrak{m}_1,k}$ is different from $B_{\mathfrak{m}_2,k}$. So there are generic points of $\mathfrak{m}_1$ in $I_{\mathfrak{m}_1,k} \cap I_{\mathfrak{m}_2,k}$. Choose a generic point p_0 for $\mathfrak{m}_1$ in $I_{\mathfrak{m}_1,k} \cap I_{\mathfrak{m}_2,k}$ which is different from $A_{\mathfrak{m}_1,k}$ and $B_{\mathfrak{m}_2,k}$. There is an admissible return g such that $g(p_0) \in V$ (recall the observation that precedes the lemma), which implies $g(p_0) \notin I_{\mathfrak{m}_2,k}$. On the other hand $p_0 \in I_{\mathfrak{m}_2,k}$ by assumption and $g(I_{\mathfrak{m}_2,k}) \subseteq I_{\mathfrak{m}_2,k}$ by (4.4.9). Since the diffeomorphisms f_i's are monotone $g(p_0) \in I_{\mathfrak{m}_2,k}$. This is a contradiction. Therefore, $I_{\mathfrak{m}_1,k}$ and $I_{\mathfrak{m}_2,k}$ have empty intersection or coincide. □

Again consider trapping domains $\mathcal{I}_{\mathfrak{m}}$ corresponding to ergodic stationary measures $\mathfrak{m}$ in $\mathcal{N}_c$. According to Lemma 4.4.7 these trapping domains are non-intersecting or coincide. By Lemma 4.4.6 for each trapping domain $\mathcal{I}_{\mathfrak{m}}$ each end point $A_{\mathfrak{m},k}$ and $B_{\mathfrak{m},k}$ which does not coincide with $x = 0$ or $x = 1$ (respectively) is an image of a fixed point of a simple return by a simple transition. On the other hand, since Ω has a finite number of symbols there is only a finite number of simple returns and simple transitions and by condition (4.2.1) in Section 4.2.1 any simple return has only finitely many fixed points. Hence, for any $k \in \Omega$ only a finite number of $I_{\mathfrak{m},k}$'s can exist in I. Therefore, we conclude that there are finitely many disjoint trapping domains and corresponding to them finitely many disjoint trapping strips for F by Proposition 4.4.5. For every stationary measure $\mathfrak{m} \in \mathcal{N}_c$ the corresponding domain $\mathcal{I}_{\mathfrak{m}} = \bigcup_{k=1}^{N} I_{\mathfrak{m},k}$ and strip $\mathcal{S}_{\mathfrak{m}} = \bigcup_{k=1}^{N} C_k^0 \times I_{\mathfrak{m},k}$ are equal to some trapping domain and trapping strip.

We thus obtain a finite number of stationary measures $\mathfrak{m}_t$, $1 \le t \le T$, with corresponding trapping domain $\mathcal{I}^t$ and trapping strip $\mathcal{S}^t$.

Step 3: Conditional measures.

We will see that inside each trapping strip $\mathcal{S}^t$, $1 \le t \le T$, there exists a unique invariant measurable graph Γ^t to which almost every point of the trapping strip is attracted. First we show that for each $1 \le t \le T$, $\mu^t = \mu_{\mathfrak{m}^t}$ has δ-measures as conditional measures along fibers inside the trapping strip $\mathcal{S}^t$, ν-almost surely. To prove the following lemma we follow [1, Theorem 1.8.4].

Lemma 4.4.8. *For every ergodic stationary probability measure $\mathfrak{m}$, the conditional measure $\mu_{\mathfrak{m},\omega}$ of $\mu_{\mathfrak{m}}$ is a δ-measure for ν-almost every $\omega \in \Sigma_{\mathcal{A}}$.*

Proof. Consider a $\mu_{\mathfrak{m}}$ and its conditional measures $\mu_{\mathfrak{m},\omega}$. Let $X_{\mathfrak{m}}(\omega)$ be the smallest median of $\mu_{\mathfrak{m},\omega}$, i.e. the infimum of all points x for which

$$\mu_{\mathfrak{m},\omega}([0,x]) \ge \frac{1}{2} \text{ and } \mu_{\mathfrak{m},\omega}([x,1]) \ge \frac{1}{2}.$$

The set of medians of $\mu_{\mathfrak{m},\omega}$ is a compact interval and $X_{\mathfrak{m}} : \Sigma_{\mathcal{A}} \to I$ is measurable. Define $C_{\mathfrak{m}}^-(\omega) := [0, X_{\mathfrak{m}}(\omega)]$ for which by definition $\mu_{\mathfrak{m},\omega}(C_{\mathfrak{m}}^-(\omega)) \ge \frac{1}{2}$. The set $C_{\mathfrak{m}}^-(\omega)$ is invariant: Since for every $i \in \Omega$, f_i is increasing for every $x_1 < x_2$ and ω we have $f_\omega(x_1) < f_\omega(x_2)$. This implies that x is a median of $\mu_{\mathfrak{m},\omega}$ if and only if $f_\omega(x)$ is a median of $f_\omega\mu_{\mathfrak{m},\omega}$. By invariance of $\mu_{\mathfrak{m}}$ we have $f_\omega\mu_{\mathfrak{m},\omega} = \mu_{\mathfrak{m},\sigma\omega}$. Hence, $X_{\mathfrak{m}}(\sigma\omega) = f_\omega(X_{\mathfrak{m}}(\omega))$ which implies $C^-_{\mathfrak{m}}(\sigma\omega) = f_\omega(C^-_{\mathfrak{m}}(\omega))$.

Because $\mu_{\mathfrak{m}}$ is ergodic and $C^-_{\mathfrak{m}}(\omega)$ is invariant $\mu_{\mathfrak{m},\omega}(C^-_{\mathfrak{m}}(\omega)) = 1$, ν-almost surely. By the same argument for $C^+_{\mathfrak{m}}(\omega) := [X_{\mathfrak{m}}(\omega), 1]$ for $\{X_{\mathfrak{m}}(\omega)\} = C^-_{\mathfrak{m}}(\omega) \cap C^+_{\mathfrak{m}}(\omega)$ we obtain $\mu_{\mathfrak{m},\omega}(\{X_{\mathfrak{m}}(\omega)\}) = 1$. Thus $\mu_{\mathfrak{m},\omega} = \delta_{X_{\mathfrak{m}}(\omega)}$ for ν-almost every $\omega \in \Sigma_{\mathcal{A}}$. $\square$

Lemma 4.4.9. *Every trapping strip contains a unique stationary measure with support contained in the trapping strip.*

Proof. Suppose there are two invariant ergodic measures $\mu_{\mathfrak{m}_1} \neq \mu_{\mathfrak{m}_2}$ for which $\mathcal{S}_{\mathfrak{m}_1} = \mathcal{S}_{\mathfrak{m}_2}$. By Lemma 4.4.8 there are measurable functions $X_{\mathfrak{m}_i} : \Sigma_{\mathcal{A}} \to I$ and $D_i \subset \Sigma_{\mathcal{A}}$ with $\nu(D_i) = 1$, for $i = 1, 2$, such that $\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}\mu^+_{\mathfrak{m}_i,\sigma^{-n}\omega} = \delta_{X_{\mathfrak{m}_i}(\omega)}$ for every $\omega \in D_i$ respectively. From $\nu(D_i) = 1$ we have $D_1 \cap D_2 \neq \emptyset$. Therefore, there is $\bar\omega \in D_1 \cap D_2$ so that $X_{\mathfrak{m}_1}(\bar\omega) \neq X_{\mathfrak{m}_2}(\bar\omega)$. Without loss of generality suppose that $X_{\mathfrak{m}_1}(\bar\omega) < X_{\mathfrak{m}_2}(\bar\omega)$.

Since $\mathcal{S}_{\mathfrak{m}_1} = \mathcal{S}_{\mathfrak{m}_2}$ we have that for every $k \in \Omega$, $I_{\mathfrak{m}_1,k} = I_{\mathfrak{m}_2,k}$. So we can find generic points $(k, x_{1,k})$ and $(k, x_{2,k})$ for $\mathfrak{m}_1$ and $\mathfrak{m}_2$ such that $x_{1,k} > x_{2,k}$. Because $f^n_{\sigma^{-n}\bar\omega}(x_{i,\bar\omega_{-n}})$ converges to $X_{\mathfrak{m}_i}(\bar\omega)$ as $n \to \infty$, and for each $j \in \Omega$, f_j is strictly increasing, we conclude that $X_{\mathfrak{m}_2}(\bar\omega) < X_{\mathfrak{m}_1}(\bar\omega)$, contradicting our assumption. Thus, $\mu_{\mathfrak{m}_1} = \mu_{\mathfrak{m}_2}$ is unique in $\mathcal{S}_{\mathfrak{m}}$. $\square$

Step 4. Attracting graphs

By Lemma 4.4.8 and 4.4.9, for every $1 \leq t \leq T$ there exists a unique measurable function $X^t : \omega \mapsto X^t(\omega)$ for each $\mathcal{S}^t$ with the domain $D^t \subset \Sigma_{\mathcal{A}}$, $\nu(D^t) = 1$, such that $\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}\mu^{t,+}_{\sigma^{-n}\omega} = \delta_{X^t(\omega)}$ for each $\omega \in D^t$. So there are graphs Γ^t of X^t with $\Gamma^t \subset \mathcal{S}^t$ which are invariant because $X^t(\sigma\omega) = f_\omega(X^t(\omega))$. Therefore, for every generic point (k, x_k) for $\mathfrak{m}^t$ we have $\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x_{\omega_{-n}}) = X^t(\omega)$. Since the fiber maps are strictly increasing for every choice of (k, x_k) with $x_k \in I_{\mathfrak{m}^t,k}$ (different from $0, 1$) and $\omega \in D^t$,

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x_{\omega_{-n}}) = X^t(\omega). \tag{4.4.10}$$

For a trapping strip $\mathcal{S}^t$ denote by A^t_ω the intersection of its maximal attractor, $A^t_{\max}$, with the fiber $I_\omega = \{\omega\} \times I$. For an $\omega \in \Sigma_{\mathcal{A}}$ define

$$A^t_{\omega,n} = f_{\omega_{-1}} \circ \cdots \circ f_{\omega_{-n}}(\mathcal{S} \cap I_{\sigma^{-n}\omega}).$$

Since the strip $\mathcal{S}^t$ is trapping for every $n \in \mathbb{N}$, $A^t_{\omega,n+1} \subseteq A^t_{\omega,n}$. Hence, for each $\omega \in \Sigma_{\mathcal{A}}$, $A^t_\omega = \bigcap_{n\geq 0} A^t_{\omega,n}$ is either an interval or a single point. So for every $\omega \in D^t$, $X(\omega) \in A^t_\omega$ and $\Gamma^t \subset A^t_{\max}$.

We state two theorems on the structure of the maximal attractor in internal and in border trapping strips, respectively. The first result is contained in [5].

Theorem 4.4.10. *Let $\mathcal{S}^t$ be an internal trapping strip of a skew product F under the assumptions of Theorem 4.2.1. Then the maximal attractor A^t_{max} is a bony graph. The attracting graph Γ^t forms the graph part of A^t_{max}.*

Theorem 4.4.11. *Let $\mathcal{S}^t$ be a border trapping strip of a skew product F under the assumptions of Theorem 4.2.1. Then the maximal attractor A^t_{max} is a thick graph. The attracting graph Γ^t in $\mathcal{S}^t$ is dense in A^t_{max}:*

$$\overline{\Gamma^t} = A^t_{max}. \tag{4.4.11}$$

Proof. We suppress the index t from the notation. Consider, without loss of generality, a border trapping strip $\mathcal{S}$ that contains $\Sigma_{\mathcal{A}} \times \{0\}$. For a trapping strip $\Sigma_{\mathcal{A}} \times [0,1]$, the maximal attractor obviously has positive standard measure. For a trapping strip with one internal boundary, since $x = 0$ is a fixed point for each f_ω and $\Gamma \subset \mathcal{S}$, we have $A_\omega = [0, X(\omega)]$, where $X(\omega) > 0$ for $\omega \in D$. So, ν-almost surely A_ω has positive Lebesgue measure and $s(A_{\max}) > 0$.

Now we prove the density of the graph Γ for border trapping strips. We restrict to the case of a trapping strip with one internal boundary. Let $\mathfrak{m}$ be the ergodic stationary measure supported in $\mathcal{S} = \mathcal{S}_{\mathfrak{m}} = \bigcup_{k=1}^N C^0_k \times I_{\mathfrak{m},k}$. By Lemma 4.4.6, there is a simple return map f with $f'(0) < 1$. Hence, for some $1 \le k_0 \le N$ there are admissible return maps $f, g : I_{\mathfrak{m},k_0} \to I_{\mathfrak{m},k_0}$ and J, a small neighborhood of $x = 0$, such that for every $k \in \Omega$, $J \subset I_{\mathfrak{m},k}$, $f'(x) < 1$ and $g'(x) > 1$ for every $x \in J$.

Lemma 4.4.12. *All orbits of the iterated function system generated by f, g restricted to J are dense in it, i.e., for any point $x \in J$ and an open interval $J' \subset J$ there is a return map $h_{k_0} \in \mathrm{IFS}\{f, g\}$ such that $h_{k_0}(x) \in J'$.*

Proof. This follows [2, Lemma 3], see Proposition 4.2.4. □

Lemma 4.4.13. *Let $\mathfrak{m}$ be an ergodic stationary measure in $\mathcal{N}_c$, k and k' be two arbitrary symbols and $I_{\mathfrak{m},k}, I_{\mathfrak{m},k'}$ with $A_{\mathfrak{m},k}, A_{\mathfrak{m},k'} = 0$ and $B_{\mathfrak{m},k}, B_{\mathfrak{m},k'} \neq 1$. Then, for any $\varepsilon > 0$ there exists an admissible composition $G : I_{\mathfrak{m},k} \to I_{\mathfrak{m},k'}$ such that $G(I_{\mathfrak{m},k}) \subset U_\varepsilon(A_{\mathfrak{m},k'})$.*

Proof. See [5, Lemma 6.9]. □

To establish (4.4.11), we need to show that for every point $P \in A_{\max}$ and every neighborhood $\mathcal{U}$ of P there exists a point Q in Γ such that $Q \in \mathcal{U}$. We may assume $\mathcal{U} = C^{-m,\dots,m}_{\hat{\omega}} \times U$, $m \in \mathbb{N}$, where $\hat{\omega} \in D$ and U is a small interval in $[0, X(\hat{\omega})]$. We need to find $\tilde{\omega} \in C^{-m,\dots,m}_{\hat{\omega}} \cap D$ such that $X(\tilde{\omega}) \in U$. Take a sequence $\omega' \in D$ with past part $\dots\omega'_{-3}\omega'_{-2}\omega'_{-1} = \omega'_-$ and denote $x' = X(\omega')$. Note that X depends only on the past part of the sequence, so x' depends on ω'_-.

By Lemma 4.4.13 there is an admissible composition $G = f_{\alpha_n} \circ \dots \circ f_{\alpha_1} : I_{\mathfrak{m},\omega'_0} \to I_{\mathfrak{m},k_0}$ so that $G(I_{\mathfrak{m},\omega'_0}) \subset J$ and $\alpha_1 = \omega'_0$. Hence, $G(x') \in J$. Let $U_m \subset J \subset I_{\mathfrak{m},k_0}$ be defined by an admissible composition such that

$$f_{\hat{\omega}_{-1}} \circ \dots \circ f_{\hat{\omega}_{-m}} \circ f_{\beta_\kappa} \circ \dots \circ f_{\beta_1}(U_m) = U,$$

where $\beta_1 = k_0$. By Lemma 4.4.12 there is a return map $h_{k_0} = f_{\eta_r} \circ \dots \circ f_{\eta_1} : I_{\mathfrak{m},k_0} \to I_{\mathfrak{m},k_0}$ in $\mathrm{IFS}\{f, g\}$ that takes $G(x')$ to U_m. Take

$$\tilde{\omega} = \omega'_- \alpha_1 \dots \alpha_n \eta_1 \dots \eta_r \beta_1 \dots \beta_\kappa \hat{\omega}_{-m} \dots \hat{\omega}_0 \dots \hat{\omega}_m \omega''_+,$$

where $\tilde{\omega}_0 = \hat{\omega}_0$ and ω''_+ is any admissible sequence. Indeed, with such a construction the sequence $\tilde{\omega}$ belongs to $C_{\hat{\omega}}^{-m,\ldots,m} \cap D$, $f^{n+r+\kappa}_{\sigma^{-(n+r+\kappa+m)}\tilde{\omega}}(x') \in U_m$ and $f^m_{\sigma^{-m}\tilde{\omega}}(U_m) \in U$. Therefore, $X(\tilde{\omega}) = f^{(n+r+\kappa+m)}_{\sigma^{-(n+r+\kappa+m)}\tilde{\omega}}(x') \in U$ and $Q = (\tilde{\omega}, X(\tilde{\omega})) \in \Gamma \cap \mathcal{U}$. □

4.4.2 An attracting end point

It remains to consider cases with negative Lyapunov exponents at end points, i.e. where $L(0) < 0$ or $L(1) < 0$ or both. Note that in an internal trapping strip or border trapping strip bounded by an end point with positive Lyapunov exponent, stationary measures are constructed as before and the analysis proceeds as in the previous sections.

The following subcases remain:

1. $L(0)$ and $L(1)$ have different signs and F has no internal trapping strip,
2. $L(0)$ and $L(1)$ are both negative and F has no internal trapping strip,
3. at least one of $L(0)$ or $L(1)$ is negative and F admits an internal trapping strip.

$L(0)$ and $L(1)$ have different signs and F has no internal trapping strip.

Here $L(0)$ and $L(1)$ have different signs. To be definite, say $L(0) > 0$ and $L(1) < 0$. We claim that the only ergodic stationary measures are point measures $\mathfrak{d}^0$, $\mathfrak{d}^1$ on 0 and 1; $\mathfrak{d}^0_k = p_k\delta_0$ and $\mathfrak{d}^1_k = p_k\delta_1$ on the intervals I_k. Indeed, let $\mathfrak{m}$ be a stationary measure that assigns mass outside the points $\{0, 1\}$. Suppose that the convex hull of the support of $\mathfrak{m}$ is a union of intervals $[A_k, 1] \subset I_k$. This defines a trapping strip that we denote by $\mathcal{S}$. As in the previous section, the stationary measure $\mathfrak{m}$ gives rise to an attracting invariant graph Γ of a map $X : D \to [0, 1]$ with $\nu(D) = 1$, so that Γ lies inside $\mathcal{S}$ and for every $(\sigma^{-n}\omega, x_{\omega_{-n}}) \in \mathcal{S}$, $\omega \in D$ (and $x_i \neq 0, 1$),

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x_{\omega_{-n}}) = X(\omega) \tag{4.4.12}$$

(compare (4.4.10)). Since the skew product system has negative Lyapunov exponent at the endpoint 1, $\Sigma_{\mathcal{A}} \times \{1\}$ has a basin of attraction with positive standard measure. This contradicts (4.4.12). Hence, there is no stationary measure $\mathfrak{m}$ that assigns mass outside the points $\{0, 1\}$. It follows that for almost all $\omega \in \Sigma$,

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x) = 1$$

for any $x \in (0, 1]$.

$L(0)$ and $L(1)$ are both negative and F has no internal trapping strip.

By the reasoning in the previous section, the inverse skew product map admits an attracting invariant graph. The skew product system hence has a repelling invariant graph, and attracting invariant graphs $\Sigma_{\mathcal{A}} \times \{0\}$, $\Sigma_{\mathcal{A}} \times \{1\}$.

At least one of $L(0)$ or $L(1)$ is negative and F admits an internal trapping strip.

Suppose $L(0) < 0$. Again by following the reasoning in the previous section, the inverse skew product map then admits a border trapping strip, bounded by $\Sigma_{\mathcal{A}} \times \{0\}$, that contains an attracting invariant graph. The skew product system hence has a repelling invariant graph, and an attracting invariant graph $\Sigma_{\mathcal{A}} \times \{0\}$.

Bibliography

[1] L. Arnold. *Random dynamical systems.* Springer-Verlag, 1998.

[2] Y. S. Ilyashenko. Thick Attractors of Step Skew products. *Regular and Chaotic Dynamics* 15, 328–334, 2010.

[3] Y. S. Ilyashenko. Thick Attractors of boundary preserving diffeomorphisms. *Indag. Math.* 22, 257–314, 2011.

[4] I. Kan. Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin. *Bull. Amer. Math. Soc.* 31, 68–74, 1994.

[5] V. Kleptsyn, D. Volk. *Physical measures for nonlinear random walks on interval. Mosc. Math. J.* 14, 339–365, 2014.

[6] Yu. G. Kudryashov. Bony attractors. *Funct. Anal. Appl.* 44, 219–222, 2010.

Summary

This thesis contains three related articles which are inserted in separate chapters. The chapters are self contained, hence it is possible to read each of them independently. In each of the chapters we discuss iterated function systems, IFSs, generated by finitely many maps $f_1, \ldots, f_k$ on the unit interval $I = [0,1]$. This means that we take random compositions of the maps f_i: for each iterate pick a symbol i at random, independently from previous iterates and with fixed probability $p_i > 0$, and then apply the map f_i. Given a sequence of outcomes $\omega_0, \ldots, \omega_{n-1}$, we thus find a composition $f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}$ which is denoted by f^n_ω.

For a study of dynamics we collect all sequences of symbols $1, 2, \ldots, k$ in the product set $\Sigma_k^+ = \{1, \ldots, k\}^{\mathbb{N}}$ and define $F^+ : \Sigma_k^+ \times [0,1] \to \Sigma_k^+ \times [0,1]$ by

$$F^+(\omega, x) = (\sigma\omega, f_{\omega_0}(x)).$$

Here $\sigma : \Sigma_k^+ \to \Sigma_k^+$ is the shift operator; $(\sigma\omega)_i = \omega_{i+1}$ for $\omega = (\omega_i)_0^\infty$. These skew product systems provide a setting to study all possible compositions of the maps $f_1, \ldots, f_k$ in a single framework. Indeed, for initial conditions $(\omega, x) \in \Sigma_k^+ \times [0,1]$, the coordinate in $[0,1]$ iterates as

$$x,\ f_{\omega_0}(x),\ f_{\omega_1} \circ f_{\omega_0}(x),\ f_{\omega_2} \circ f_{\omega_1} \circ f_{\omega_0}(x),\ \ldots$$

One can also consider two sided sequences of symbols in $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$ and the map $F : \Sigma_k \times [0,1] \to \Sigma_k \times [0,1]$ given by the same formula

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x))$$

as F^+. A natural measure on Σ_k^+ is the product measure ν^+, given probabilities p_i for symbols i. On Σ_k one likewise defines a product measure, which we denote by ν. These measures are invariant under the shift, so for instance $\nu^+(A) = \nu^+(\sigma^{-1}(A))$ for all measurable sets.

As an illustrative example, in Chapter 2 we consider C^2-diffeomorphisms f_1 and f_2 on I that fulfill the following conditions:

1. $f_i(0) = 0,\ f_i(1) = 1$ for $i = 1, 2$;
2. $f_1(x) < x$ for $x \in (0,1)$;
3. $f_2(x) > x$ for $x \in (0,1)$.

Both diffeomorphisms f_1, f_2 are picked with probability 1/2. Note these maps fix the boundaries of the interval I. The sign of the Lyapunov exponent at a boundary determines whether the boundary is, on average, attracting (negative Lyapunov exponent), neutral (zero), or repelling (positive).

In general, intermittency in a dynamical system stands for dynamics that exhibits alternating phases of different characteristics. In our context, we say that the skew product system F^+ displays intermittency if the following holds for any sufficiently small neighborhood U of 0:

1. For all $x \in (0,1)$ and ν^+-almost all $\omega \in \Sigma_2^+$:

$$\lim_{n\to\infty} \frac{1}{n} \left|\{0 \le i < n \; ; \; f_\omega^i(x) \in U\}\right| = 1;$$

2. For all $x \in (0,1)$ and ν^+-almost all $\omega \in \Sigma_2^+$, $f_\omega^n(x) \notin U$ for infinitely many n.

Here, for a finite set S its cardinality is written by $|S|$. Assuming zero Lyapunov exponent at 0 and positive Lyapunov exponent at 1, we prove in Theorem 2.5.8 that in this situation the iterated function system displays intermittency.

Going back to the general setting, we note that invariant measures are central in the study of the skew product systems. Invariant product measures $\nu^+ \times m$ for F^+ are of particular importance. The measure m here is a stationary measure for the Markov process that is defined by the iterated function system and the given probabilities. Given a stationary measure m, the skew product system F admits a corresponding invariant measure μ.

Let us provide a second example from Chapter 3 in which the iterated function system is given by finitely many logistic maps

$$f_i(x) = \rho_i x(1-x),$$

with $0 < \rho_i \leqslant 4$. By synchronization we mean that under identical compositions of logistic maps orbits of different initial conditions converge to each other. Assuming the boundary 0 is repelling on average and providing some conditions involving negative Lyapunov exponents and minimal dynamics we prove that synchronization occurs. More formally we assume that for the iterated function system of logistic maps IFS $(\{f_1, \ldots, f_k\})$,

1. $\rho_i \neq 4$;

2. For some $1 \leqslant i_1 \leqslant k$, the map f_{i_1} possesses an attracting fixed point in $(0,1)$ with basin of attraction equal to $(0,1)$;

3. The Lyapunov exponent at 0 is positive;

4. There is an ergodic stationary probability measure m such that

   (a) with respect to m, the iterated function system has negative Lyapunov exponents;

(b) the iterated function system is minimal on $\operatorname{supp}(m) \setminus \{0\}$.

Then we prove that for ν^+-almost all $\omega \in \Sigma_k^+$ there is an open set $W^s(\omega) \subset I$ with $m(W^s(\omega)) = 1$ so that for $x, y \in W^s(\omega)$,

$$\lim_{n\to\infty} |f_\omega^n(x) - f_\omega^n(y)| = 0.$$

See Theorem 3.3.1 in Chapter 3.

For the proof we combine the existence of a large stable set for the logistic map f_{i_1} with the use of a pullback convergence argument. To apply this the extension to the two sided shift on Σ_k is needed. If μ is the invariant measure for F corresponding to m, then the conditional measure μ_ω on $\{\omega\} \times I$ is a delta measure for ν-almost all ω: there exists a measurable function $X : \Sigma_k \to I$ so that

$$\lim_{n\to\infty} f_{\sigma^{-n}\omega}^n m = \delta_{X(\omega)},$$

for ν-almost all ω, with convergence in the weak star topology.

Samenvatting

Dit proefschrift is samengesteld uit drie gerelateerde artikelen die verschillende hoofdstukken vormen. Deze hoofdstukken kunnen afzonderlijk gelezen worden. Elk van de hoofdstukken gaat over geïtereerde functiesystemen gegenereerd door eindig veel afbeeldingen $f_1, \ldots, f_k$ op het eenheidsinterval $I = [0, 1]$. Dit betekent dat willekeurige samenstellingen van de afbeeldingen worden genomen: voor iedere iteratie wordt een index i gekozen, bijvoorbeeld uit een vaste kansverdeling onafhankelijk van de vorige keuze, waarna f_i wordt toegepast. Een rij $\omega_0, \ldots, \omega_{n-1}$ van uitkomsten geeft dus een samenstelling $f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}$. We schrijven dit ook als f_ω^n.

Voor een studie van de dynamica verzamelen we alle mogelijke rijen symbolen $1, \ldots, k$ in de productruimte $\Sigma_k^+ = \{1, \ldots, k\}^{\mathbb{N}}$ en definiëren we $F^+ : \Sigma_k^+ \times I \to \Sigma_k^+ \times I$ door

$$F^+(\omega, x) = (\sigma\omega, f_{\omega_0}(x)).$$

Hier staat $\sigma : \Sigma_k^+ \to \Sigma_k^+$ voor de schuifafbeelding; $(\sigma\omega)_i = \omega_{i+1}$ voor $\omega = (\omega_i)_0^\infty$. Deze scheve productsystemen geven een zetting om alle mogelijke samenstellingen van de afbeeldingen $f_1, \ldots, f_k$ is één raamwerk te bestuderen. Namelijk, voor een beginwaarde $(\omega, x) \in \Sigma_k^+ \times I$, vinden we iteraties

$$x, f_{\omega_0}(x), f_{\omega_1} \circ f_{\omega_0}(x), f_{\omega_2} \circ f_{\omega_1} \circ f_{\omega_0}(x), \ldots$$

voor de coördinaat in I. We kunnen ook tweezijdig oneindige rijen symbolen in $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$ nemen, met de afbeelding $F : \Sigma_k \times [0, 1] \to \Sigma_k \times [0, 1]$ gegeven door dezelfde formule

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x))$$

als F^+. Een natuurlijke waarschijnlijkheidsmaat op Σ_k^+ is de productmaat ν^+ met gegeven kansen p_i voor de symbolen i. Op Σ_k hebben we de productmaat ν. Deze maten zijn invariant onder de schuifafbeelding σ: zo geldt $\nu^+(A) = \nu^+(\sigma^{-1}(A))$ voor meetbare verzamelingen.

Neem als voorbeeld, uit het tweede hoofdstuk, twee gladde inverteerbare afbeeldingen f_1, f_2 op I die voldoen aan

1. $f_i(0) = 0$, $f_i(1) = 1$ for $i = 1, 2$;

2. $f_1(x) < x$ for $x \in (0, 1)$;

3. $f_2(x) > x$ for $x \in (0,1)$.

Beide afbeeldingen worden gekozen met kans 1/2. Merk op dat beide afbeeldingen de randpunten $0, 1$ van I vast laten. Of de randpunten gemiddeld aantrekkend, neutraal, of afstotend zijn wordt bepaald door de Lyapunov exponenten die dan negatief, nul, of positief zijn.

In zijn algemeenheid staat intermittentie voor dynamica waarin tijdreeksen afwisselend verschillende karakteristieken laten zien. In onze context heet de dynamica van het scheve productsysteem F^+ intermittent als voor iedere kleine omgeving U van 0 het volgende geldt:

1. Voor alle $x \in (0,1)$ en ν^+-bijna alle $\omega \in \Sigma_2^+$:
$$\lim_{n\to\infty} \frac{1}{n} \left|\{0 \le i < n\ ;\ f_\omega^i(x) \in U\}\right| = 1;$$

2. Voor alle $x \in (0,1)$ en ν^+-bijna alle $\omega \in \Sigma_2^+$, $f_\omega^n(x) \notin U$ voor oneindig veel waarden van n.

Voor een eindige verzameling S schrijven we hier $|S|$ voor het aantal elementen in S. We bewijzen dat intermittentie optreedt in het geval van een Lyapunov exponent die nul is in 0 en een Lyapunov exponent die positief is in 1.

In de algemene zetting geldt dat invariant maten centraal staan in de studie van scheve productsystemen. Invariante productmaten $\nu^+ \times m$ voor F^+ zijn van bijzonder belang. De maat m hierin is namelijk een stationaire maat voor het Markov proces dat gegeven wordt door de afbeelingen $f_1, \ldots, f_k$ en de kansen waarmee die afbeeldingen gekozen worden. Bij een stationaire maat m hoort een invariante maat μ voor F.

We beschouwen een tweede systeem uit het derde hoofdstuk van dit proefschrift waarin eindig veel logistische afbeeldingen
$$f_i(x) = \rho_i x(1-x),$$
$0 < \rho_i \le 4$, bekeken worden. In dit hoofdstuk ligt de nadruk op synchronizatie: het effect dat banen onder identieke samenstellingen van de logistische afbeeldingen naar elkaar convergeren. We bewijzen dat synchronizatie optreedt als er een positieve Lyapunov in 0 is en er aan voorwaarden over negatieve Lyapunov exponenten en minimaliteit van de dynamica in het open interval $(0,1)$ voldaan is. Meer formeel, neem aan dat de volgende voorwaarden op de afbeeldingen $f_1, \ldots, f_k$ gelden.

1. $\rho_i \neq 4$;

2. Voor een $1 \leqslant i_1 \leqslant k$ bezit f_{i_1} een aantrekkend vast punt in $(0,1)$ met bassin $(0,1)$;

3. De Lyapunov exponent in 0 is positief;

4. Er is een ergodische stationaire maat m zodat

(a) met betrekking tot m, het geïteeerde functiesysteem een negatieve Lyapunov exponent heeft;

(b) het geïtereerde functiesysteem minimaal is op $\operatorname{supp}(m) \setminus \{0\}$.

We laten zien dat onder deze voorwaarden, voor ν^+-bijna alle $\omega \in \Sigma_k^+$, er een open verzameling $W^s(\omega) \subset I$ is met $m(W^s(\omega)) = 1$ en zodat voor alle $x, y \in W^s(\omega)$,

$$\lim_{n \to \infty} |f_\omega^n(x) - f_\omega^n(y)| = 0.$$

Author contributions

This thesis contains one submitted and two published articles:

Chapter 2: M. Gharaei, A.J. Homburg. Random interval diffeomorphisms. *Discrete Contin. Dyn. Syst. Ser. S* (10), 241-272, 2017.

Chapter 3: N. Abbasi, M. Gharaei, A.J. Homburg. Iterated function systems of logistic maps: synchronization and intermittency, preprint, 2017.

Chapter 4: M. Gharaei, A.J. Homburg. Skew products of interval maps over subshifts. *J. Difference Equ. Appl.* 22, 941-958, 2016.

Topics and design followed from intensive discussions between the involved authors. MG contributed in a substantial and significant way to all aspects of the research, from design and drafting proofs to final editing, for all the papers.

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